J. Fluid Mech. (1998), *vol.* 368, *pp.* 195–228. Printed in the United Kingdom

Faraday's instability in viscous fluid

By E. A. CERDA[1,2]† AND E. L. TIRAPEGUI[2,3]

[1]Dept. de Física de la Universidad de Santiago de Chile, Av. Ecuador 3493, Casilla 307-2, Santiago, Chile

[2]Centro de Física No Lineal y Sistemas Complejos de Santiago, Casilla 27122, Santiago, Chile

[3]Dept. de Física de la Fac. de Ciencias Físicas y Matemáticas de la Universidad de Chile, Beaucheff 850, Casilla 487-3, Santiago, Chile

(Received 31 March 1997 and in revised form 2 March 1998)

We find a quantitative approximation which explains the appearance and amplification of surface waves in a highly viscous fluid when it is subjected to vertical accelerations (Faraday's instability). Although stationary surface waves with frequency equal to half of the frequency of the excitation are observed in fluids of different kinematical viscosities we show here that the mechanism which produces the instability is very different for a highly viscous fluid as compared with a weakly viscous fluid. This is achieved by deriving an exact equation for the linear evolution of the surface which is non-local in time. We show that for a highly viscous fluid this equation becomes local and of second order and is then a Mathieu equation which is different from the one found for weak viscosity. Analysing the new equation we find an intimate relation with the Rayleigh–Taylor instability.

1. Introduction

When a layer of fluid is vertically accelerated stationary waves are observed with half the frequency of the excitation (Faraday's phenomena). In the case of weakly viscous fluids (such as water) one has surface waves with a dispersion relation expressing their frequency ω_k in terms of the wavenumber k. If such a fluid is forced with a frequency Ω the surface waves with frequency and wavenumber given by the relation $\omega_k = \Omega/2$ are selected through Faraday's mechanism. This last relation has been used in the past to obtain information about the fluid (Eisenmerger 1959) since the wavelength of the excited waves can be measured for any frequency Ω of excitation and in this way one has access to the dispersion relation. A first explanation of the mechanism underlying the instability and of the generation of stationary waves was proposed in different contexts by Benjamin & Ursell (1954) and by Sorokin (1957). It was shown that the amplitude of the free surface for a wavenumber k obeys a Mathieu equation which is the equation for an harmonic oscillator with a periodic frequency (see for example Bechhoefer *et al.* 1995 for a review).

In the last decade Faraday's instability has been extensively investigated for weakly viscous fluids in the framework of nonlinear phenomena. Some examples are the appearance of structures near the onset of the instability in confined and in extended systems (Wu, Keolian & Rudnick 1984; Douady & Fauve 1988; Douady 1990; Elphick & Meron 1989; Edwards & Fauve 1992, 1993, 1994; Zhang & Viñals 1996,

† Present address: MIT, 3-357, 77, Mass. Ave. Cambridge, MA 02139, USA, email: ecerda@mit.edu.

1997; Chen & Viñals 1997); secondary instabilities and transition to spatio-temporal chaos (e.g. Ciliberto & Gollub 1985; Kudrolli & Gollub 1996); and turbulence (Pushkarev & Zakharov 1996; Schröder *et al.* 1996). Several works have also been devoted to the study of the instability in strongly dissipative systems (for example oil) both experimentally and theoretically (Müller 1993; Kumar & Tuckerman 1994; Beyer & Friedrich 1995; Bechhoefer *et al.* 1995; Kumar 1996; Kudrolli & Gollub 1996; Besson, Edwards & Tuckerman 1996; Lioubashevski, Arbell & Fineberg 1996; Lioubashevski, Fineberg & Tuckerman 1997). The theoretical works have been mainly numerical studies of the Navier–Stokes (NS) equation which have predicted with great accuracy two important measurable quantities: the threshold of the acceleration needed to observe the instability and the wavelength of the associated structure. These works show that the NS description is adequate to describe Faraday's instability in a highly viscous fluid, but they leave open questions about the mechanism which produces the stationary waves. This is quite puzzling if one thinks that in a fluid like oil waves relax very rapidly and disappear if no forcing is present: we are facing then a phenomena which depends crucially on the injection of energy. On the other hand one can measure in experiments the wavelength of the stationary waves as a function of the excitation frequency Ω and obtain a dispersion relation just as for a weakly viscous fluid, but one can ask now what is the meaning of this dispersion relation if the waves are overdamped in the absence of forcing. The answer to these questions can also be relevant to the understanding of experiments in granular matter where one also observes Faraday's instability (Melo, Umbanhowar & Swinney 1994, 1995; Umbanhowar, Melo & Swinney 1996), i.e. the appearance of stationary waves with half the frequency of the excitation. In this last case it is even more obvious that the waves do not exist without forcing and the measured dispersion relation suggests the same questions.

In order to understand what happens in a highly viscous fluid we have derived an exact equation for the linear evolution of the amplitude of the free surface. This equation was first derived in Cerda & Tirapegui (1997*a*) but only a few of its implications were considered there. We present here a much more complete study. The derivation and physical interpretation of the equation is discussed in § 2. With this equation we can analyse different regimes going from a weakly viscous fluid to a highly viscous one which is our main interest here. In § 3 we find a Mathieu equation for weak viscosity in agreement with the results of Benjamin & Ursell (1954) and Sorokin (1957) but what is more surprising is that for high viscosity we also find a Mathieu equation. In § 4 we make a simplified analysis of our new Mathieu equation and in § 5 we make an analytical study which gives a clear understanding of the numerical results and also experimental results. Furthermore our analysis leads to a physical interpretation which relates Faraday's instability to the Rayleigh–Taylor instability.

2. The equation for the surface

2.1. *General formalism*

We consider a fluid layer of height h and choose $\hat{z}$ as the unitary vector in the vertical direction with $z = 0$ corresponding to the surface of the fluid at rest and in contact with the atmosphere and $z = -h$ corresponding to the position of the plate (see figure 1). Due to the symmetry of the problem it is convenient to separate the coordinates in vertical z and horizontal $\boldsymbol{x} = (x, y)$ directions. The velocity of the fluid

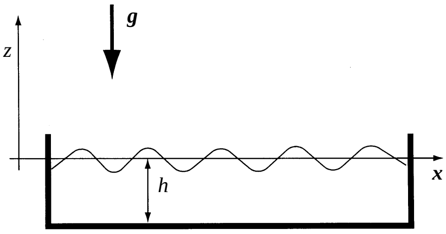

FIGURE 1. Sketch of the experiment. The origin of the system of coordinates is on the surface at rest.

is described by the field $\boldsymbol{v}(\boldsymbol{x},z,t)$ which we assume incompressible ($\nabla\cdot\boldsymbol{v} = 0$) and obeying the NS equation

$$\partial_t\boldsymbol{v}(\boldsymbol{x},z,t) + (\boldsymbol{v}(\boldsymbol{x},z,t)\cdot\nabla)\boldsymbol{v}(\boldsymbol{x},z,t) = -\frac{1}{\rho}\nabla p(\boldsymbol{x},z,t) - g_e(t)\hat{z} + \nu\nabla^2\boldsymbol{v}(\boldsymbol{x},z,t); \tag{2.1}$$

here ρ is the mass density, p is the pressure, ν the kinematic viscosity and $g_e(t)$ is the effective gravity, i.e. gravity g plus the acceleration of the plate. We call $\xi(\boldsymbol{x},t)$ the displacement of the surface from $z = 0$. The appropriate boundary conditions on the free surface are the kinematic condition (from now on we write $\boldsymbol{v}_\perp = (v_x, v_y)$ for the horizontal velocity and $\nabla_\perp = (\partial x, \partial y)$)

$$v_z(\boldsymbol{x},z,t)|_{z=\xi} = \partial_t\xi(\boldsymbol{x},t) + (\boldsymbol{v}_\perp(\boldsymbol{x},\xi,t)\cdot\nabla_\perp)\xi(\boldsymbol{x},t), \tag{2.2}$$

and the condition expressing the equality of the forces

$$T_{jk}\hat{n}_k|_{z=\xi} = [p_0 + \tau(1/R_1 + 1/R_2)]\hat{n}_j, \tag{2.3}$$

where $T_{jk} = p\delta_{jk} - \rho\nu(\partial_j v_k + \partial_k v_j)$ is the stress tensor, $\hat{n}$ the unit outward normal to the surface, p_0 the atmospheric pressure which we assume constant, τ the surface tension and R_1, R_2 are the radius of curvature of the surface calculated using the principal axes (see Landau & Lifshitz 1987). On the plate we have the no-slip boundary condition

$$\boldsymbol{v}(\boldsymbol{x},z,t)|_{z=-h} = 0. \tag{2.4}$$

In the absence of viscosity one has to solve Euler's equation with the boundary conditions implied by the absence of tangential stress and in this case we have a potential flow $\boldsymbol{v} = -\nabla\phi$ as solution of the equations. When dissipation is included we must solve the equations written above and one can check that the potential flow satisfying Euler's equation is also a solution of the NS equation since $\nu\nabla^2\boldsymbol{v} = -\nu\nabla(\nabla^2\phi) = 0$ but this solution cannot satisfy the boundary conditions for a viscous fluid and one expects that the motion ceases to be potential near the boundaries, i.e. in the free surface and on the plate. One defines then a length δ which characterizes the size of this boundary layer where the motion is non-potential in order to satisfy the correct boundary conditions. It is well-known that for weak viscosity this characteristic length decreases with viscosity as $\delta \sim \nu^{1/2}$. From the previous discussion it seems reasonable to separate the velocity into two components, $\boldsymbol{v} = -\nabla\phi + \boldsymbol{u}$, with both terms incompressible, in such a way that the component $\boldsymbol{u}$ is only relevant near the boundaries while the potential part is important in the rest of the fluid.

We solve equations (2.1)–(2.4) taking as reference state the solution in which the free surface is flat (rest solution) which is

$$\boldsymbol{v}_{st}(\boldsymbol{x},z,t) = 0, \quad p_{st}(t) = p_0 - \rho g_e(t)z.$$

We now rewrite the original equations putting $p = \rho\pi + p_{st}$ and keeping only linear terms in $\boldsymbol{v}$ and π, the NS equation becomes

$$(\partial_t - \nu\nabla^2)\boldsymbol{v}(\boldsymbol{x}, z, t) = -\nabla\pi(\boldsymbol{x}, z, t). \tag{2.5}$$

Taking the divergence of (2.5) and using incompressibility one has $\nabla^2\pi = 0$ which we solve and then insert the solution as a source term in (2.5). We look for a solution in the form of a sum of a particular solution $\boldsymbol{v}_{par}$ and the general solution $\boldsymbol{u}$ of the homogeneous equation. A particularly convenient $\boldsymbol{v}_{par}$ is obtained putting $\boldsymbol{v}_{par} = -\nabla\phi$ and demanding incompressibility, $\nabla^2\phi = 0$, which gives $\partial_t\phi = \pi$ for the potential ϕ which is analogous to the Bernoulli equation. The NS equation and the incompressibility condition reduce to the equivalent set

$$\nabla^2\phi(\boldsymbol{x}, z, t) = 0, \tag{2.6a}$$

$$\partial_t\phi(\boldsymbol{x}, z, t) = \pi(\boldsymbol{x}, z, t), \tag{2.6b}$$

$$(\partial_t - \nu\nabla^2)\boldsymbol{u}(\boldsymbol{x}, z, t) = 0; \tag{2.6c}$$

the last equation is the homogeneous part of (2.5) and the component $\boldsymbol{u}$ will be called for obvious reasons the diffusive component of the velocity which is also incompressible by construction. We write down now the linear boundary conditions. The kinematic condition is

$$v_z(\boldsymbol{x}, z, t)|_{z=0} = \partial_t\xi(\boldsymbol{x}, t), \tag{2.7}$$

and the equality of forces in equation (2.3) can be separated into a part normal to the surface

$$(\pi(\boldsymbol{x}, z, t) - 2\nu\partial_z v_z(\boldsymbol{x}, z, t))|_{z=0} = g_e(t)\xi(\boldsymbol{x}, t) - \frac{\tau}{\rho}\nabla_\perp^2\xi(\boldsymbol{x}, t), \tag{2.8}$$

and another part related to the tangential directions

$$(\partial_x v_z(\boldsymbol{x}, z, t) + \partial_z v_x(\boldsymbol{x}, z, t))|_{z=0} = 0, \tag{2.9a}$$

$$(\partial_y v_z(\boldsymbol{x}, z, t) + \partial_z v_y(\boldsymbol{x}, z, t))|_{z=0} = 0. \tag{2.9b}$$

The condition on the plate remains the same. Since several boundary conditions are expressed only in terms of v_z one can try to obtain a simpler independent problem for this component. Conditions (2.7)–(2.8) are already in the required form. Differentiating (2.9*a*) with respect x, (2.9*b*) with respect y and adding both equations taking into account the incompressibility condition we obtain

$$(\nabla_\perp^2 - \partial_z^2)v_z(\boldsymbol{x}, z, t)|_{z=0} = 0.$$

Summarizing, the boundary conditions in the free surface can be written in terms of the velocity potential ϕ and of the diffusive velocity u_z as

$$(u_z(\boldsymbol{x}, z, t) - \partial_z\phi(\boldsymbol{x}, z, t))|_{z=0} = \partial_t\xi(\boldsymbol{x}, t), \tag{2.10}$$

$$(\pi(\boldsymbol{x}, z, t) - 2\nu\partial_z u_z(\boldsymbol{x}, z, t) + 2\nu\partial_z^2\phi(\boldsymbol{x}, z, t))|_{z=0} = g_e(t)\xi(\boldsymbol{x}, t) - \frac{\tau}{\rho}\nabla_\perp^2\xi(\boldsymbol{x}, t), \tag{2.11}$$

$$(\nabla_\perp^2 - \partial_z^2)(u_z(\boldsymbol{x}, z, t) - \partial_z\phi(\boldsymbol{x}, z, t))|_{z=0} = 0, \tag{2.12}$$

where everything has been expressed in terms of ϕ, π, ξ and u_z. Since u_z satisfies the diffusion equation we can rewrite (2.12) in the form

$$(\partial_t - 2\nu\nabla_\perp^2)u_z(\boldsymbol{x}, z, t)|_{z=0} + \nu(\nabla_\perp^2 - \partial_z^2)\partial_z\phi(\boldsymbol{x}, z, t)|_{z=0} = 0$$

and due to $\nabla^2\phi = 0$ one has

$$\partial_t u_z(\boldsymbol{x}, z, t)|_{z=0} - 2\nu\nabla_\perp^2(u_z(\boldsymbol{x}, z, t) - \partial_z\phi(\boldsymbol{x}, z, t))|_{z=0} = 0,$$

then we can replace in the last term the boundary condition (2.10) and integrate over time with no constants of integration since the rest solution $u_z = \xi = 0$ must be a solution of the problem. We obtain

$$u_z(\boldsymbol{x}, z, t)|_{z=0} = 2\nu\nabla_\perp^2\xi(\boldsymbol{x}, t)$$

and then equations (2.10)–(2.12) can be replaced by an equivalent set of conditions given below in (2.14*a*–*c*). Finally the boundary condition on the plate (2.4) can also be expressed in terms of the vertical component of the velocity as follows: using the incompressibility relation $\partial_z v_z|_{z=-h} = -\nabla_\perp \cdot \boldsymbol{v}_\perp|_{z=-h} = 0$ where the second equality is due to the vanishing of the tangential components and consequently one has on the plate

$$v_z|_{z=-h} = 0, \quad \partial_z v_z|_{z=-h} = 0. \tag{2.13}$$

The linear problem is now reduced to solving equations (2.6*a*–*c*) with the following boundary conditions: on the surface one must satisfy

$$(u_z(\boldsymbol{x}, z, t) - \partial_z\phi(\boldsymbol{x}, z, t))|_{z=0} = \partial_t\xi(\boldsymbol{x}, t), \tag{2.14a}$$

$$(\pi(\boldsymbol{x}, z, t) - 2\nu\partial_z u_z(\boldsymbol{x}, z, t) + 2\nu\partial_z^2\phi(\boldsymbol{x}, z, t))|_{z=0} = g_e(t)\xi(\boldsymbol{x}, t) - \frac{\tau}{\rho}\nabla_\perp^2\xi(\boldsymbol{x}, t), \tag{2.14b}$$

$$u_z(\boldsymbol{x}, z, t)|_{z=0} = 2\nu\nabla_\perp^2\xi(\boldsymbol{x}, t), \tag{2.14c}$$

and on the plate one has

$$(u_z(\boldsymbol{x}, z, t) - \partial_z\phi(\boldsymbol{x}, z, t))|_{z=-h} = 0, \tag{2.15a}$$

$$\partial_z(u_z(\boldsymbol{x}, z, t) - \partial_z\phi(\boldsymbol{x}, z, t))|_{z=-h} = 0. \tag{2.15b}$$

We remark that at this level no assumptions have been made with respect to lateral boundary conditions and we might well include them. This point is discussed by Edwards & Fauve (1993) and Bechhoefer *et al.* (1995). The effect of the lateral walls may be important for several reasons: a surface wave can travel to a wall and interact with it or the wall can generate capillary waves which can contaminate the system. However when dissipation is strong enough the surface waves generated by the walls are damped in a short distance and the effect of the walls can be neglected.

2.2. *A self-contained problem for the surface*

In order to solve equations (2.6*a*–*c*) we neglect the effect of lateral walls and we make a Fourier transform in the horizontal coordinates for all the variables. We can solve Laplace equation (2.6) in the form

$$\phi_{\boldsymbol{k}}(z, t) = A_{\boldsymbol{k}}(t)\cosh k(z+h) + B_{\boldsymbol{k}}(t)\sinh k(z+h)$$

and using all the boundary conditions except (2.14*b*) we obtain

$$\phi_{\boldsymbol{k}}(z, t) = -\frac{(\partial_t + 2\nu k^2)\xi_{\boldsymbol{k}}(t)}{k\sinh kh}\cosh k(z+h) - \frac{u_{z\boldsymbol{k}}(z, t)|_{z=-h}}{k\sinh kh}\cosh kz. \tag{2.16}$$

We now use condition (2.14*b*) to obtain an equation for the amplitude $\xi_{\boldsymbol{k}}$ of the surface corresponding to the mode k. We find

$$(\partial_t + 2\nu k^2)^2\xi_{\boldsymbol{k}}(t) + \omega_k^2(t)\,\xi_{\boldsymbol{k}}(t) + \frac{(\partial_t + 2\nu k^2)}{\cosh kh}u_{z\boldsymbol{k}}(z, t)|_{z=-h} + 2\nu k\tanh kh\,\partial_z u_{z\boldsymbol{k}}(z, t)|_{z=0} = 0, \tag{2.17}$$

where $\omega_k^2(t) \equiv k\,(g_e(t) + \tau k^2/\rho)\tanh kh$ reduces to the usual frequency ω_k of surface waves when $g_e(t) = g$. Equation (2.17) is an exact equation for the motion of the surface of a viscous fluid. One must of course also provide an appropriate problem for the variable $u_{z\boldsymbol{k}}$ which appears in (2.17). The equation and boundary conditions for $u_{z\boldsymbol{k}}$ are easily obtained from the other equations and they are

$$[\partial_t - \nu(\partial_z^2 - k^2)]u_{z\boldsymbol{k}}(z,t) = 0, \tag{2.18}$$

$$u_{z\boldsymbol{k}}(z,t)|_{z=0} = -2\nu k^2 \xi_{\boldsymbol{k}}(t), \tag{2.19}$$

$$[\sinh kh\,\partial_z u_{z\boldsymbol{k}}(z,t) + k\cosh kh\,u_{z\boldsymbol{k}}(z,t)]|_{z=-h} = -k(\partial_t + 2\nu k^2)\xi_{\boldsymbol{k}}(t). \tag{2.20}$$

Solving this problem, i.e. equations (2.18)–(2.20), one obtains an expression for $u_{z\boldsymbol{k}}$ which is a functional of $\xi_{\boldsymbol{k}}$, and inserting this expression in (2.17) we finally obtain a self-contained equation for the surface amplitude $\xi_{\boldsymbol{k}}$ which was our ultimate aim. Equation (2.17) is much more complex than the corresponding equation in the non-dissipative case but each new term has a clear physical interpretation: (*a*) the effect of both boundaries can be clearly identified in the last two terms of this equation, the first one corresponding to the bottom ($z = -h$ on the plate) and the second one to the upper boundary (the free surface at $z = 0$), and (*b*) the additional contribution which is a translation in the temporal derivative ($\partial_t \to \partial_t + 2\nu k^2$) can be identified as the contribution to dissipation due to friction in the region of potential motion of the fluid. These interpretations are developed further in the following section.

2.3. *Interpretation of the equation*

In order to get more insight into the interpretation of equation (2.17) we derive now from it some well-known results of fluid mechanics for constant gravity $g_e(t) = g$. Without dissipation ($\nu = 0$) one obtains from equation (2.17) that the surface obeys the equation for a harmonic oscillator (from now on we shall also use the notation $\dot{f} = \partial_t f$)

$$\ddot{\xi}_{\boldsymbol{k}}(t) + \omega_k^2\,\xi_{\boldsymbol{k}}(t) = 0, \tag{2.21}$$

which has surface waves with frequency ω_k corresponding to solutions $\xi_{\boldsymbol{k}} = \xi_0 \mathrm{e}^{\mathrm{i}\omega_k t}$ (the complex conjugate is also a solution and the same discussion applies to it due to linearity). In the weakly viscous case (the meaning of this is taken now just as effects in lowest order of the viscosity) the previous solution is the dominant part of the amplitude and it allows us to find the behaviour of the diffusive velocity using equations (2.18)–(2.20). This problem is very similar to the Stokes problem (see Landau & Lifshitz 1987) in which one has a fluid over an oscillating wall. In our case we obtain

$$u_{z\boldsymbol{k}}(z,t) = \left[-2\nu k^2 \mathrm{e}^{z/\delta}\mathrm{e}^{\mathrm{i}z/\delta} + \frac{(1+\mathrm{i})k\delta\omega_k}{2\sinh kh}\mathrm{e}^{-(z+h)/\delta}\mathrm{e}^{-\mathrm{i}(z+h)/\delta}\right]\xi_0\mathrm{e}^{\mathrm{i}\omega_k t}, \tag{2.22}$$

where $\delta = (2\nu/\omega_k)^{1/2}$. The interpretation of this result is well-known: the solution for the velocity profile is non-potential only near the boundaries, i.e. the free surface and the plate. One can also observe that the effect of the lower surface is not important when $kh \gg 1$ and the reason for this is that according to (2.16) the motion of the surface penetrates the fluid a distance ℓ which we can estimate as

$$\ell \approx \begin{cases} k^{-1}, & kh \gg 1 \\ h, & kh \lesssim 1. \end{cases} \tag{2.23}$$

We see then that when $kh \gg 1$ the fluid in motion is not in contact with the plate and due to this $u_{z\boldsymbol{k}}$ does not contribute there. The length ℓ will appear recurrently in this paper since is the relevant quantity to discuss easily important limits such as shallow water or deep water. For example a criterion to say that a fluid is weakly dissipative is to ask whether the region of non-potential behaviour of the fluid is small, and since the size of this region is of $O(\delta)$ the appropriate way to express the condition is to compare δ with ℓ which is the size of the region where the fluid is in motion and to write $\delta \ll \ell$.

Let us consider now the damping of a surface wave. If we neglect the effect of the boundaries in our general equation (2.17) we have

$$(\partial_t + 2\nu k^2)^2 \xi_{\boldsymbol{k}}(t) + \omega_k^2\, \xi_{\boldsymbol{k}}(t) = 0; \tag{2.24}$$

this equation tells us that the amplitude of the wave is damped as $\xi_0 = \text{const.} \times \mathrm{e}^{-2\nu k^2 t}$ which is the same damping factor one obtains by other methods (Landau & Lifshitz 1987) and whose interpretation is of damping by friction in the bulk of the fluid. This effect is caused by the term $(\partial_t + 2\nu k^2)$ in (2.17) which is a translation of the temporal derivative. To be more precise about the contribution of the different terms to the evolution of a surface wave we consider the energy $\epsilon_{\boldsymbol{k}}$ of a wave

$$\epsilon_{\boldsymbol{k}}(t) = \tfrac{1}{2}(|\dot{\xi}_{\boldsymbol{k}}|^2 + \omega_k^2 |\xi_{\boldsymbol{k}}|^2). \tag{2.25}$$

Without dissipation this energy is conserved; when one adds viscosity as in equation (2.17) this energy decreases in time. Using the profile (2.22) one can determine the terms which dominate this decrease:

$$\dot{\epsilon}_{\boldsymbol{k}} = -4\nu k^2 |\dot{\xi}_{\boldsymbol{k}}|^2 - \frac{1}{2\cosh kh}(\dot{\xi}^*_{\boldsymbol{k}} \partial_t u_{z\boldsymbol{k}}|_{z=-h} + \dot{\xi}_{\boldsymbol{k}} \partial_t u^*_{z\boldsymbol{k}}|_{z=-h}) + O\left((\nu k^2 \omega_k)^{3/2} |\xi_k|^2\right). \tag{2.26}$$

We observe that at this order in the viscosity the effect of the upper boundary is negligible. For weak viscosity the amplitude ξ_0 of an oscillation decays slowly and we can average (2.26) over one period of the oscillation after putting $\xi_{\boldsymbol{k}} = \xi_0 \mathrm{e}^{i\omega_k t}$ and using the velocity profile (2.22). The result is

$$\langle \dot{\epsilon}_{\boldsymbol{k}} \rangle = -\left(4\nu k^2 + \frac{k\delta\omega_k}{\sinh 2kh}\right) \omega_k^2 |\xi_0|^2, \tag{2.27}$$

and in the same way one has that $\langle \epsilon_{\boldsymbol{k}} \rangle = \omega_k^2 |\xi_0|^2$ and then the energy evolves as

$$\langle \dot{\epsilon}_{\boldsymbol{k}} \rangle = -2\gamma \langle \epsilon_{\boldsymbol{k}} \rangle, \qquad \gamma = 2\nu k^2 + \frac{k\delta\omega_k}{2\sinh 2kh}. \tag{2.28}$$

Since the energy decays as $\sim \mathrm{e}^{-2\gamma t}$ and is proportional to the square of the amplitude ξ_0, the latter decays as $\sim \mathrm{e}^{\gamma t}$ where γ is known as the damping coefficient for surface waves. The first term of the expression for γ in (2.28) is produced by the translation of the temporal derivative which we have identified with dissipation in the bulk and the second term has its origin in the lower boundary (the plate) and is easy to interpret since it vanishes when $kh \gg 1$, i.e. when the fluid in motion has no contact with the plate. Furthermore this latter contribution behaves as $\gamma \sim \nu^{1/2}$ which is characteristic of dissipation by friction of a fluid in contact with an oscillating solid surface (as in the Stokes problem) where in our case it is the fluid which oscillates, and the conclusion is then that the term represents dissipation by friction between the fluid and the plate in the bottom.

2.4. *The exact solution for* $g_e(t) = g$

It is possible to solve exactly equations (2.17)–(2.20) when $g_e(t) = g$. We give here only a brief summary of our results since a complete discussion can be found in Cerda & Tirapegui (1997*b*). Using the results of Appendix A we obtain for the velocity profile the solution

$$u_{z\boldsymbol{k}}(z,t) = \int_{t_0}^{\infty} \mathrm{d}t'\xi_{\boldsymbol{k}}(t')\mathrm{e}^{-\nu k^2(t-t')}K(t-t',z) + \int_{-h}^{0} \mathrm{d}z' u_{z\boldsymbol{k}}(z',t_0)\mathrm{e}^{-\nu k^2(t-t_0)}G(t-t_0,z',z), \tag{2.29}$$

where the kernels K and G are given in the Appendix. This solution for $u_{z\boldsymbol{k}}$ is the same when one has forcing and when it is replaced in our equation (2.17) the result is an independent equation for the surface which is however non-local in time. This fact has several consequences: in the absence of forcing we find for weak viscosity that in addition to the two usual frequencies λ_+ and λ_- (without viscosity $\lambda_\pm = \pm\mathrm{i}\,\omega_k$) one has an infinite number of new frequencies $\{\lambda_m\}$ and consequently the amplitude of the surface can be written in the form

$$\xi_{\boldsymbol{k}}(t) = A_+\mathrm{e}^{\lambda_+(t-t_0)} + A_-\mathrm{e}^{\lambda_-(t-t_0)} + \sum_{m=1}^{\infty} A_m \mathrm{e}^{\lambda_m(t-t_0)}, \tag{2.30}$$

where A_+, A_- and $\{A_m\}$ are constants and t_0 is an initial time. Consistently with the non-viscous limit in which the behaviour is given by the frequencies λ_+ and λ_- the amplitudes $\{A_m\}$ vanish when $\nu \to 0$.

The infinite number of frequencies is due to the property of non-locality in time of the equation for $\xi_{\boldsymbol{k}}$ which can be exhibited by writing this equation formally with time derivatives of all orders since a dispersion relation consistent with the non-locality must contain an infinite set of frequencies. We obtain the dispersion relation (see also Kumar 1996)

$$DR(s) = (s+2\nu k^2)^2 + \omega_k^2 + \nu^2 k \frac{(q^2+k^2)}{\cosh kh}\left(\frac{4qk\sinh kh - (q^2+k^2)\sinh qh}{k\sinh qh\cosh kh - q\sinh kh\cosh qh}\right) + 4\nu^2 qk^3 \tanh kh \left(\frac{q\sinh qh\sinh kh - k\cosh qh\cosh kh}{k\sinh qh\cosh kh - q\sinh kh\cosh qh}\right), \tag{2.31}$$

where $q = (s/\nu + k^2)^{1/2}$. The zeros of $DR(s)$ as a function of s are the frequencies λ_+, λ_- and $\{\lambda_m\}$. Due to the complexity of (2.31) we rewrite it in a more useful way by means of a Mittag–Lefler expansion (Markushevich 1970) which exhibits explicitly the singularities of the dispersion relation which are in all cases simple poles at

$$\tilde{s}_m = -\nu k^2 - \nu p_m^2, \tag{2.32}$$

where $\{p_m\}$ are defined in Appendix A. Since all the numbers p_m are real the poles $\tilde{s}_m$ are in the negative real axis and $DR(s)$ has the form

$$DR(s) = s^2 + a_0 s + \omega_k^2 + s^2 \sum_{m=1}^{\infty} \frac{a_m}{s - \tilde{s}_m}, \tag{2.33}$$

where the coefficients $\{a_0, a_1, \ldots\}$ are all positive with values

$$a_m = \begin{cases} \dfrac{4\nu k^2}{(k^2+p_m^2)^2} \dfrac{\left[(k^2-p_m^2)\sin p_m h - 2p_m k \sinh kh\right]^2}{\sinh 2kh(kh - \tanh kh \cos^2 p_m h)}, & m \neq 0 \\ -2\nu k^2 \tanh kh \dfrac{((kh)^2 + \cosh^2 kh)}{(kh - \tanh kh \cosh^2 kh)}, & m = 0. \end{cases} \tag{2.34}$$

Using expression (2.33) we can show that the frequencies $\{\lambda_1, \lambda_2, \ldots, \}$ are real and negative for any values of the parameters and can be ordered as $\tilde{s}_1 > \lambda_1 > \tilde{s}_2 > \lambda_2 > \tilde{s}_3, \ldots$, and we see then that each of these frequencies represents an aperiodic damping.

3. Faraday's instability

3.1. *Results without dissipation ($\nu = 0$)*

We consider now the case with forcing where one observes surface waves with half the frequency of the forcing, i.e. Faraday's instability. Before treating the high viscosity case which is our main interest here, we review the classical weak viscosity limit which will give us a good starting point for the discussion of the high viscous limit. From (2.17) we see that without dissipation the equation reduces to

$$\ddot{\xi}_{\boldsymbol{k}}(t) + \omega_k^2(t)\xi_{\boldsymbol{k}}(t) = 0, \tag{3.1}$$

where $\omega_k^2(t) = k\,(g_e(t) + \tau k^2/\rho)\tanh kh$ includes the forcing. This equation was derived by Benjamin & Ursell (1954) who gave an explanation for the instability. For a typical forcing of the plate given by the acceleration $\ddot{z}_p = a\cos\Omega t$ the effective acceleration is $g_e(t) = g(1 + \Gamma\cos\Omega t)$ where the parameter $\Gamma = a/g$ represents the amplitude of the acceleration of the plate in units of gravity. Equation (3.1) can be written as a Mathieu equation (Landau & Lifshitz 1970)

$$\ddot{\xi}_{\boldsymbol{k}}(t) + \omega_k^2(1 + \Gamma_k \cos\Omega t)\xi_{\boldsymbol{k}}(t) = 0, \tag{3.2}$$

with $\Gamma_k = \Gamma/(1 + \tau k^2/g\rho)$. This is the equation of a parametric oscillator for which a typical example is a pendulum whose point of suspension has a periodic motion. In order to use this analogy one must take into account that in the our case we have a pendulum of frequency ω_k in which the effect of gravity g is displaced to $g + \tau k^2/\rho$ and then the pendulum feels a 'stronger gravity' which increases with the wavenumber k. In fact the correct comparison between accelerations is not through Γ but rather through Γ_k and this is precisely what one finds in equation (3.2). Since both Γ_k and the frequency ω_k of surface waves are functions of the wavenumber k it is convenient to use the (Γ, k) space in the study of the Mathieu equation. Numerically one obtains the series of tongues of figure 2 and the interiors of these tongues correspond to unstable regions. It can be verified that for $\Gamma_k \ll 1$ the instability occurs whenever the wavenumber satisfies the resonance condition (analytical studies can be found in Landau & Lifshitz 1970; Bergé, Pomeau & Vidal 1984)

$$\omega_k = n\frac{\Omega}{2} \quad \text{with} \quad n = 1, 2, \ldots, \infty. \tag{3.3}$$

This resonance condition allows one to classify each tongue according to the value of n in (3.3). For odd n the solution of (3.2) corresponding to a point inside the tongue changes its phase by π during one period of the forcing and due to this property these tongues are called subharmonic. For even n there is no change in the phase after one period and these tongues are called harmonic. We can see in the figure that

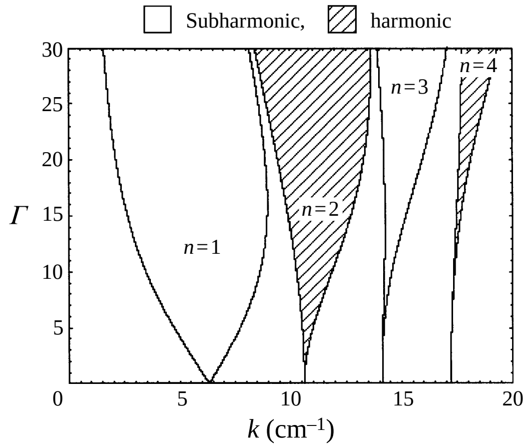

FIGURE 2. Unstable tongues in the absence of dissipation. Data is for water $\tau = 74\,\mathrm{g\,s^{-2}}$, $\rho = 1\,\mathrm{g\,cm^{-3}}$, $g = 980\,\mathrm{cm\,s^{-2}}$, $h = 1\,\mathrm{cm}$ and the frequency of forcing is $\Omega/2\pi = 50\,\mathrm{Hz}$. When the resonance condition is satisfied the corresponding mode k is unstable.

there is no minimal threshold Γ to produce the instability: as soon as the forcing acts an infinite number of modes are excited and it is necessary to include dissipation in order to understand which mode is selected. As remarked by Benjamin & Ursell (1954) dissipation favours selection of the first resonance $n = 1$ and the reason is that other resonances correspond to modes with bigger wavenumber and consequently with stronger spatial gradients which favour dissipation and increase the threshold.

3.2. *Incorporation of dissipation: the phenomenological approximation*

If dissipation is taken into account one has to use our general equation (2.17) which is non-local in time as discussed in §2 and this fact complicates the obtainment of analytical results. Our point of view here is to look for regions in the space of parameters in which equation (2.17) admits as a good approximation an equation which is local in time and hence in principle much more tractable, in order to have the possibility of making a complete study of the instability. For example when viscosity is weak, i.e. $\delta \ll \ell$, one can expect the existence of a local equation since the strict limit $\nu = 0$ corresponds to equation (3.1) which is local. Before continuing we recall that the length δ accounts for the penetration inside the fluid of the effect of the boundaries and it is given according to our discussion in §2.3 by $\delta = (2\nu/\omega_k)^{1/2}$; on the other hand we have stated that the most unstable mode when dissipation is included is inside the first subharmonic tongue which corresponds to $\omega_k \approx \Omega/2$, hence it is more convenient to define this length as $\delta = (\nu/\Omega)^{1/2}$ which has the advantage of expressing δ in terms of experimental parameters. With respect to ℓ one must use the definition given in §2.3 taking the mode k which is selected, i.e. the one that is determined by the wavelength effectively measured in an experiment or the one that results from a theoretical calculation of the selected mode (if the correct equation is used!).

In the weakly viscous case it would be reasonable to try an expansion in the dimensionless parameter δ/ℓ, but such a perturbation expansion is singular. This singular character can be seen by studying directly the origin of the non-locality in time: if one observes that a perturbation in the diffusive velocity takes a characteristic time $\tau_d \sim \ell^2/\nu$ to propagate in the fluid layer we can say that at time t the equation for the surface should take into account information about the fluid up to a time

$t-\tau_d$, and this means in fact that we are identifying τ_d with the characteristic time of the memory effects. Consequently the memory effects, i.e. the non-locality in time, are present if τ_d is big when we compare it with the relevant characteristic time of the system which is Ω^{-1}, i.e. if $\tau_d \gg \Omega^{-1}$. This last condition is equivalent to $\delta \ll \ell$ which is the condition for weak viscosity and we can conclude that the non-local character of the equation for the surface is more relevant when the viscosity is weak.

Although a systematic derivation does not exist a way to incorporate the dissipative effects in the equation (3.1) for the surface is to consider the situation of deep water eliminating the dissipative effect in the bottom, i.e. on the plate (see Sorokin 1957). We observed in § 2.2 that in such a situation the dissipation by friction in the bulk of the fluid is $O(\nu)$ while the effect of the upper boundary is $O(\nu^{3/2})$, and we see then that the contribution to dissipation of the bulk is dominant. At $O(\nu)$ it is then enough to consider the following equation:

$$\ddot{\xi}_{\boldsymbol{k}}(t) + 2\gamma_k \dot{\xi}_{\boldsymbol{k}}(t) + \omega_k^2(t)\, \xi_{\boldsymbol{k}}(t) = 0, \tag{3.4}$$

where the damping coefficient is $\gamma_k = 2\nu k^2$. This approximation can be obtained from our generalized equation (2.17) through an *ad-hoc* truncation of the non-local terms which does not constitute a systematic derivation. An example of the difficulties one has to face due to the non-locality appears when $kh \lesssim 1$ since in this case the contribution to dissipation of the region near the plate cannot be ignored because it is dominant for weak viscosity (it is of $O(\nu^{1/2})$ according to our discussion in § 2.2) and on the other hand this boundary term is non-local in time with a memory τ_d which is long for $\delta \ll \ell$; hence a local approximation as in deep water is doubtful.

The effect of dissipation in equation (3.4) is the expected one since it increases with the wavenumber k and has then the consequence of increasing the threshold of the forcing when k increases, a fact which can be observed in figure 3: the most unstable tongue is the first subharmonic tongue; all the others are higher and consequently the observed mode in experiments has frequency $\omega_k \approx \Omega/2$. One should remark that this equation is derived under the assumption $k\delta \ll 1$ and one can check that this condition is equivalent to $\gamma_k \ll \Omega$ which is a convenient way to state that the oscillator described by equation (3.4) is weakly damped (the current definition in the literature is $\gamma_k \ll \omega_k$ which is equivalent due to the resonance condition). An analytic calculation can be done in such a case for the threshold of the first subharmonic tongue with the result $\Gamma_k \approx 4\gamma_k/\omega_k$ (see Landau & Lifshitz 1970), i.e. the threshold is determined by dissipation in the bulk. The phenomenological model has been extensively studied and compared with numerical simulations of the original NS equation which account exactly for the stability problem; this technique developed by Kumar (1996) and Kumar & Tuckerman (1994) leads to the conclusion that when the viscosity is weak enough and when the stated assumptions are verified the phenomenological model is quantitatively correct for the prediction of the threshold Γ and of the observed wavelength. Since equation (3.4) is simple enough to allow an efficient discussion of dissipative effects it has been widely used (see for example Edwards & Fauve 1992; Pushkarev & Zakharov 1996).

3.3. *The strongly dissipative case: lubrication approximation*

When δ is greater than or of the order as ℓ ($\delta \gtrsim \ell$) the system is strongly dissipative. The meaning is that the boundary layer saturates the region in which the motion of the fluid occurs, i.e. the size of the boundary layer is approximately ℓ; the length $\delta = (\nu/\Omega)^{1/2}$ can be even greater than ℓ and this indicates that one should not interpret δ anymore as the size of the boundary layer and that it is necessary to

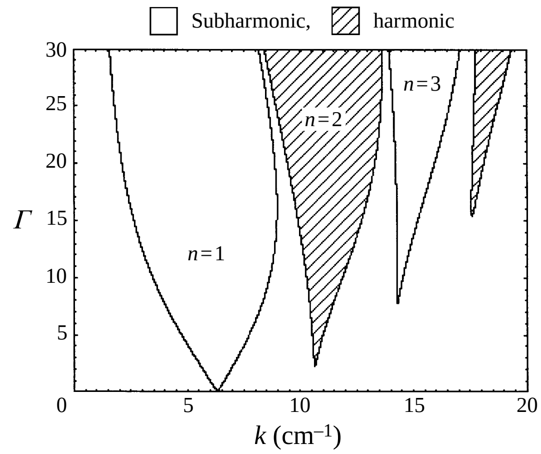

FIGURE 3. Unstable tongues when a small damping effect is included phenomenologically. We use $\nu = 0.01\,\mathrm{cm^2\,s^{-1}}$ for water. The first subharmonic tongue is the most unstable (the threshold Γ is very low and is not observed in the figure).

reinterpret the condition $\delta \stackrel{\sim}{>} \ell$. From the discussion of § 3.2 we can conclude that the previous condition is much more a comparison between times than a comparison of lengths: the memory in our non-local equation has been identified with the diffusion time $\tau_d = \ell^2/\nu$ and then stating that $\delta \stackrel{\sim}{>} \ell$ is equivalent to saying that $\tau_d \stackrel{\sim}{<} \Omega^{-1}$, and the physical meaning is that we are in a region of parameters in which diffusive effects occur 'rapidly'. This has an important consequence: since the memory time is now comparatively short an approximation local in time to our general equation (2.17) for the surface must exist and we call it the 'lubrication approximation' since it is related to similar approaches in strongly dissipative systems (see for example de Gennes 1985).

The lubrication approximation can be derived systematically as an expansion of our general equation for the surface (2.17) in the parameter $\tau_d\Omega$ and it turns out that each new term in the expansion represents a temporal derivative of higher order. Since we are only interested in the stability problem we eliminate transients in the velocity profile (2.29) taking the limit $t_0 \to -\infty$. We obtain

$$u_{z\boldsymbol{k}}(z,t) = \int_{-\infty}^{\infty} \mathrm{d}t'\,\xi_{\boldsymbol{k}}(t')\mathrm{e}^{-\nu k^2(t-t')}K(t-t',z), \tag{3.5}$$

where the kernel $K(t-t',z)$ satisfies causality and vanishes then for $t-t'<0$ and also when $t-t'\to\infty$ since the memory effects are of finite duration (see Appendix A). Putting $\sigma = t-t'$ we write (3.5) in the form

$$u_{z\boldsymbol{k}}(z,t) = \int_{-\infty}^{\infty} \mathrm{d}\sigma\,\xi_{\boldsymbol{k}}(t-\sigma)\mathrm{e}^{-\nu k^2\sigma}K(\sigma,z), \tag{3.6}$$

where now the integration time σ represents the delay needed to determine the velocity profile at time t. If we place in (3.6) the formal expansion

$$\xi_{\boldsymbol{k}}(t-\sigma) = \sum_{m\geqslant 0}\frac{(-\sigma)^m}{m!}\partial_t^m\xi_{\boldsymbol{k}}(t) \tag{3.7}$$

we obtain for the velocity profile an expansion in derivatives of the amplitude $\xi_{\boldsymbol{k}}$ of the surface:

$$u_{z\boldsymbol{k}}(z,t) = \sum_{m\geqslant 0} \frac{A_m(z)}{m!} \partial_t^m \xi_{\boldsymbol{k}}(t), \tag{3.8a}$$

$$A_m(z) = \int_{-\infty}^{\infty} \mathrm{d}\sigma (-\sigma)^m \mathrm{e}^{-\nu k^2 \sigma} K(\sigma, z). \tag{3.8b}$$

The coefficient A_m can be easily calculated in terms of the Laplace transform $\tilde{K}(s,z)$ of the kernel $K(\sigma,z)$ with respect to the time σ. One has

$$\begin{aligned} \mathrm{e}^{-\nu k^2\sigma} K(\sigma,z) &= \int_\Gamma \frac{\mathrm{d}s}{2\pi\mathrm{i}}\, \mathrm{e}^{(s-\nu k^2)\sigma} \tilde{K}(s,z) = \int_{\Gamma'} \frac{\mathrm{d}s}{2\pi\mathrm{i}}\, \mathrm{e}^{s\sigma} \tilde{K}(s+\nu k^2, z) \\ &= \sum_{p\geqslant 0} \left.\frac{\partial_s^p \tilde{K}(s,z)}{p!}\right|_{s=\nu k^2} \int_{\Gamma'} \frac{\mathrm{d}s}{2\pi\mathrm{i}}\, \mathrm{e}^{s\sigma} s^p \\ &= \sum_{p\geqslant 0} \left.\frac{\partial_s^p \tilde{K}(s,z)}{p!}\right|_{s=\nu k^2} \frac{\mathrm{d}^p}{\mathrm{d}\sigma^p} \delta(\sigma), \end{aligned} \tag{3.9}$$

where the path Γ' is obtained by displacing to the left by νk^2 the path Γ of Appendix A, figure 11, and $\delta(\sigma)$ is the Dirac distribution. Using this identity and the formula $\int_{-\infty}^{\infty} \mathrm{d}\sigma(-\sigma)^m (\mathrm{d}^p/\mathrm{d}\sigma^p)\delta(\sigma) = m!\delta_{mp}$ in the expression (3.8) for the coefficients A_m one has

$$A_m(z) = \partial_s^m \tilde{K}(s,z)\big|_{s=\nu k^2}. \tag{3.10}$$

We have now an explicit formula for the velocity profile which can be used in our general equation (2.17) which can then be written formally as the infinite series

$$\sum_{m\geqslant 1} \frac{C_m(kh)}{(\nu k^2)^{m-2}}\, \partial_t^m \xi_{\boldsymbol{k}}(t) + \omega_k^2(t)\xi_{\boldsymbol{k}}(t) = 0; \tag{3.11}$$

we remark that there are no new contributions to the last term in (3.11) which is the only one which depends on the forcing. The functions $C_m(y)$ are dimensionless functions of $y = kh$. For $m = 1$, 2 the explicit expressions which will be important in what follows are

$$C_1(y) = 2\tanh y\,(\cosh 2y + 2y^2 + 1)/(\sinh 2y - 2y),$$
$$C_2(y) = \tanh y\,[3\cosh^2 y\,(\sinh 2y - 2y - 4y^3/3) + y^2(\sinh 2y - 2y)]/(\sinh 2y - 2y)^2;$$

both functions are positive and we draw them in figure 4. The functions $C_m(y)$ have interesting properties which can be proved using the form (2.33) of the dispersion relation since the function $DR(s)$ expanded in a power series in s must coincide with the series obtained from (3.11) after the replacement $\xi_{\boldsymbol{k}}(t) = \text{const.} \times \mathrm{e}^{st}$. Using this identity we can prove that

$$C_m(y\to\infty) \sim \alpha_m, \quad C_m(y\to 0) \sim \beta_m\, y^{2(m-2)}, \tag{3.12}$$

where α_m and β_m are numerical coefficients depending on m. The effect of the shallow water limit $y \to 0$ is to replace the scale k^{-1} in the expansion (3.11) by h, i.e. the pertinent length scale in all cases is indeed the penetration of the fluid motion ℓ and the consequence of this is that the expansion can be roughly written in the form

$$\sum_{m\geqslant 1} \text{coeff} \times \tau_d^{m-2} \partial_t^m \xi_{\boldsymbol{k}}(t) + \omega_k^2(t)\xi_{\boldsymbol{k}}(t) = 0, \tag{3.13}$$

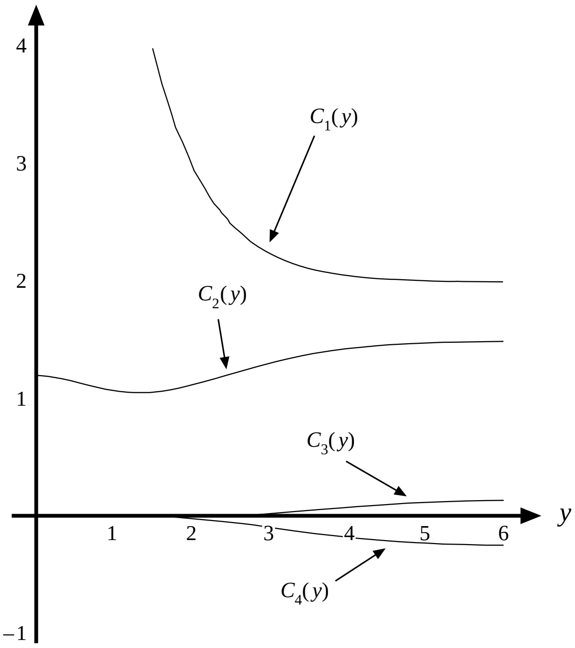

FIGURE 4. Plots of the first dimensionless functions $C_1(y)$, $C_2(y)$, $C_3(y)$ and $C_4(y)$.

where the coefficients are not important now (they are α_m or β_m in the limit cases). When the fluid undergoes a forcing of frequency Ω this introduces a characteristic time scale in the sense that we assume that the temporal derivatives are of order $\partial_t \sim O(\Omega)$. The justification for this point is twofold: the experimental results show that the system oscillates with a frequency $\Omega/2$ imposed by the forcing and on the other hand the assumption is self-consistent in the sense that the analytical solutions we find have that behaviour. Putting $x = \Omega t$ in the expression (3.13) it can be written in the dimensionless form

$$\sum_{m\geqslant 1} \text{coeff} \times (\Omega\tau_d)^{m-2}\partial_x^m \xi_{\boldsymbol{k}}(x) + \frac{\omega_k^2(x)}{\Omega^2}\xi_{\boldsymbol{k}}(x) = 0, \tag{3.14}$$

which shows explicitly that the expansion is a series in the parameter $\Omega\tau_d$. When $\Omega\tau_d \ll 1$ we can keep the most singular terms of the expansion which is given by the terms up to $m = 2$ and we obtain then a second-order equation which is local in agreement with the general arguments given first. When $\Omega\tau_d \approx 1$ the truncation can only be justified through an analysis of the numerical coefficients and it can be shown using the dispersion relation (2.33) that they are favourable for a truncation especially when $kh \lessapprox 1$ since in this case they decay exponentially with m, i.e. coeff $\sim \mathrm{e}^{-\lambda m}$ where λ is a constant. For example in the limit case $kh \ll 1$ one has $\beta_3 = -1/175$, $\beta_4 = 2/7875$, etc. When $kh \gg 1$ the coefficients decay more slowly as coeff $\sim 1/m^{3/2}$ and one has $\alpha_3 = -1/4$, $\alpha_4 = 5/32$ etc. Summarizing, in the strongly dissipative case $\Omega\tau_d \lessapprox 1$ it is sufficient to study a local equation obtained by truncating (3.11) at $m = 2$ but one must be careful if the mode selected by this equation is such that $kh \gg 1$ since it follows from our discussion that in this case the agreement could be only qualitative.

Although it is possible to study any type of forcing with our equation we shall consider here the usual forcing $g_e(t) = g(1 + \Gamma \cos \Omega t)$ and we rewrite in this case the equation truncated at $m = 2$ in the Mathieu form

$$\ddot{\xi}_{\boldsymbol{k}}(t) + 2\bar{\gamma}_k \dot{\xi}_{\boldsymbol{k}}(t) + \bar{\omega}_k^2(1 + \Gamma_k \cos \Omega t)\xi_{\boldsymbol{k}}(t) = 0 \tag{3.15}$$

and in what follows we shall refer to this new equation as the Cerda–Tirapegui (CT) equation. The new coefficients are

$$\bar{\gamma}_k = \nu k^2 \frac{C_1(kh)}{2\,C_2(kh)}, \quad \bar{\omega}_k^2 = \frac{\omega_k^2}{C_2(kh)}. \tag{3.16}$$

We have once more the equation of a parametric oscillator but now with frequency $\bar{\omega}_k$ and dissipation $\bar{\gamma}_k$. In the case of deep water this equation is similar to the phenomenological equation for weak viscosity (3.4) and the dissipation increases with the wavenumber since $C_1(y \gg 1) \approx 2$. In the opposite case of shallow water the dissipation saturates to a value which is independent of the wavenumber since in this case $C_1(y \ll 1) \approx 3/2\,y^2$. Notice that this dependence on the wavenumber is not altered by the function $C_2(y)$ since this function is always of $O(1)$ as is shown in figure 4.

Furthermore there is a qualitative difference between equations (3.4) and (3.15) since one can in general estimate $\bar{\gamma}_k \sim 1/\tau_d$ and we can then rewrite the condition of validity of the CT equation $\Omega\tau_d \lesssim 1$ as $\bar{\gamma}_k \gtrsim \Omega$. If one follows now the discussion after equation (3.4) the previous condition can be seen as a natural extension expressing that the CT equation represents a strongly damped oscillator.

4. A naive analysis of the lubrication approximation

Equation (3.15) can be studied with high precision using the WKB approximation (Cerda & Tirapegui 1997*a*) but only a naive analysis is presented here, which will be justified by the WKB approximation in § 5.

Because the dissipative coefficient $\bar{\gamma}_k$ can be non-restrictively large, for example by increasing the viscosity, the only possibility for unstable solutions in equation (3.15) is a balance between the dissipative term and the forcing term. As we shall discuss below the time derivatives can be estimated as a multiple of the driving frequency, i.e. $\partial_t \xi_{\boldsymbol{k}} \sim n\frac{1}{2}\Omega\xi_{\boldsymbol{k}}$, and then the balance is

$$n\frac{\Omega}{2}\bar{\gamma}_k \sim \bar{\omega}_k^2 \Gamma_k. \tag{4.1}$$

Since $\bar{\gamma}_k$ is large, we must also have $\Gamma_k \gg 1$ in order to verify the balance (4.1). When $\Gamma_k \gg 1$ new phenomena arise:

(i) If the analogy with a pendulum is used, the pendulum would feel negative acceleration when $\cos\Omega t < 0$. This situation is evidently unstable since the fluid feels a negative acceleration as happens when a heavier fluid is over a lighter one in which case the system is immediately unstable (Rayleigh–Taylor instability, Chandrasekar 1981). The mechanism here is then the analogue of the Rayleigh–Taylor instability as we have pointed out in Cerda & Tirapegui (1997*a*). This conclusion was also reached in Lioubashevsky *et al.* (1997) and was supported by an *ad-hoc* calculation.

(ii) There is an additional new important frequency in the equation, $\bar{\omega}_k\Gamma_k^{1/2}$. When $\cos\Omega t > 0$ the pendulum oscillates at this effective frequency. Furthemore, there is a new resonance condition because it is natural for the envelope of the instability that the amplification due to negative acceleration happens when the pendulum has a maximum amplitude. The new resonance condition is then $\bar{\omega}_k\Gamma_k^{1/2} \sim n\frac{1}{2}\Omega$, which can be written as

$$\bar{\gamma}_k \sim n\frac{\Omega}{2}, \tag{4.2}$$

because of the relation (4.1).

The function $\bar{\gamma}_k$ always increases with the wavenumber k and consequently the resonance or frequency associated to the smallest wavenumber is $n = 1$, the first subharmonic tongue. This observation is fundamental to understanding why the first resonance is observed. From relations (4.1) and (4.2) we can obtain the threshold of the instability

$$\Gamma_k \sim (\bar{\gamma}_k/\bar{\omega}_k)^2. \tag{4.3}$$

This relation can be compared with the similar expression for weakly viscous fluids, in which it is found that the threshold increases according $\Gamma_k \sim \gamma_k/\omega_k$. The threshold of Γ given by the relation (4.3) is a very interesting function of the wavenumber k (see figure 5): it has a minimum value Γ_* for the value k_* of the wavenumber. Therefore, when the resonant modes in relation (4.2) are bigger than k_* the minimum threshold corresponds to the resonance $n = 1$. On the other hand, this picture is different when there are resonant modes smaller than k_*: the resonance which is the closest to k_* is observed, therefore another resonances could be observed. That phenomenon was first predicted in Kumar (1996) and observed experimentally by Müller *et al.* (1997). In this latter paper an approximation method is proposed for a weakly viscous fluid which is opposite to the case of a strongly viscous fluid which is our main interest here. The calculation of Müller *et al.* (1997) leads to the appearance of a bicritical point, corresponding to the crossover from the first subharmonic resonance to the next harmonic resonance, which happens however to values of the parameters for which one cannot assure the validity of the approximation as the authors point out. Nevertheless, their result indicates that bicriticality could also occur in that region of parameters where the threshold increases as $\Gamma_k \sim \gamma_k/\omega_k$ which for small wavenumbers may increase when one considers that the dissipation γ_k contains corrections to the contributions of the bulk dissipation (for example, if we consider the contribution of the dissipation at the bottom boundary given in (2.28)).

The positions of the resonant modes are controlled by the frequency Ω in (4.2). If the frequency Ω is sufficiently high the resonance $n = 1$ has a large wavenumber, which means $k \gg k_*$, and then the dissipation γ_k can be estimated as $\gamma_k \sim \nu k^2$ (3.16). Using this estimation, we obtain from (4.2) that the most unstable wavenumber is given by $k_c \sim (\Omega/\nu)^{1/2}$. This has been observed by Edwards & Fauve (1992) while studying Faraday's instability for a fluid near its critical point. For the threshold of acceleration, which is necessary to observe the instability, a_c, we obtain from (4.3) $a_c \sim \nu^{1/2}\Omega^{3/2}$. We do not know of any experimental work which checks this scaling.

If the frequency is low, the resonance $n = 1$ is close to the mode k_* and then the properties of the instability are given by the properties of k_* and Γ_*. Using relation (4.3), we can see that $k_* \sim 1/h$ and $\Gamma_* \sim \nu^2/h^3 g$. We will show in § 5.6 that this has been observed by Lioubashevski *et al.* (1997).

5. Analytical results

5.1. *The WKB method*

In order to analyse equation (3.15) we write it as a Schrödinger equation after the change of variables $x = \Omega t$, $\Psi(x) = \xi_k(t)e^{\bar{\gamma}_k t}$, $E = \bar{\omega}_k^2 - \bar{\gamma}_k^2$, $V(x) = -V_0 \cos x$,

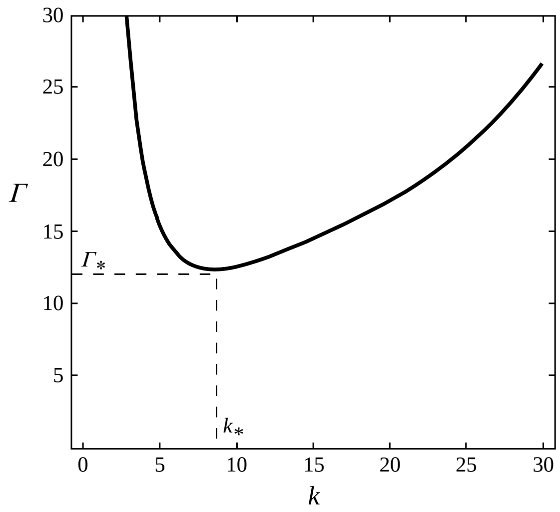

FIGURE 5. Plot of $(\bar{\gamma}_k/\bar{\omega}_k)^2(1+\tau k^2/\rho g)$ as a function of the wavenumber k. The form of the curve does not change if the other parameters are changed.

$V_0 = \Gamma_k \bar{\omega}_k^2$. We obtain

$$\Psi''(x) + \frac{1}{\Omega^2}(E - V(x))\Psi(x) = 0, \tag{5.1}$$

which is a Schrödinger equation in a periodic potential and where Ω is playing the role of $\hbar$. The WKB method can be used here since the expansion parameter of this approximation is the inverse of $(E - V(x))/\Omega^2$ and when this value is big the approximation is a very good one (Messiah 1973). A simple estimate of this parameter is

$$\left|\frac{1}{\Omega^2}(E - V(x))\right| \approx \frac{\bar{\gamma}_k^2}{\Omega^2} \approx \left(\frac{\delta}{\ell}\right)^4, \tag{5.2}$$

which is big if $\ell \ll \delta$. If $\ell \approx \delta$ the CT equation is still valid but we can only expect qualitative information from the WKB method.

The solution of this problem in the context of quantum mechanics can be found in Goldman & Krivchenkov (1961). The calculation of the WKB solutions determines the Floquet exponent μ of the solutions in physical space which are of the form

$$\xi_k(t + 2\pi/\Omega) = e^{\mu 2\pi/\Omega}\xi_k(t) \tag{5.3}$$

and the value of μ determines the stability problem. We look first for solutions $\Psi(x)$ of (5.1) with the Floquet form which have the property $\Psi(x+2\pi) = z\Psi(x)$ and then the exponent is given by

$$e^{\mu 2\pi/\Omega} = z e^{-\bar{\gamma}_k 2\pi/\Omega}. \tag{5.4}$$

One has two types of WKB solutions according to the sign of $E - V(x)$:

$$\Psi(x) = \frac{1}{(E - V(x))^{1/4}}\left(A \exp\frac{i}{\Omega}\int^x d\varphi (E - V(\varphi))^{1/2}\right.$$
$$\left. + B \exp\left(-\frac{i}{\Omega}\int^x d\varphi (E - V(\varphi))^{1/2}\right)\right), \tag{5.5}$$

if $E - V(x) > 0$ and

$$\Psi(x) = \frac{1}{(V(x)-E)^{1/4}} \left(A \exp \frac{1}{\Omega} \int^x \mathrm{d}\varphi (V(\varphi) - E)^{1/2} \right. \\ \left. + B \exp \left(-\frac{1}{\Omega} \int^x \mathrm{d}\varphi (V(\varphi) - E)^{1/2} \right) \right), \tag{5.6}$$

if $E - V(x) < 0$; A and B are arbitrary constants. It is convenient to consider three sectors in the space of parameters: $E > V_0$, $E < -V_0$ y $|E| < V_0$.

The sector $E > V_0$ (see figure 6*a*) corresponds in quantum mechanics to a particle for which one has no classically forbidden regions and the solutions are of the type (5.5), i.e. oscillatory. A solution of the Floquet form can be found taking $A \neq 0$ and $B = 0$ or vice versa. In the first case we obtain

$$z = \exp \left(\frac{\mathrm{i}}{\Omega} \int_0^{2\pi} \mathrm{d}\varphi (E - V(\varphi))^{1/2} \right), \tag{5.7}$$

$$\mu = -\bar{\gamma}_k + \frac{\mathrm{i}}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi (E - V(\varphi))^{1/2}, \tag{5.8}$$

and if $A = 0$ and $B \neq 0$ the exponent is the complex conjugate of the previous one. Since in one period the amplitude only changes its phase and is damped we have a stable situation.

When $E < -V_0$ (see figure 6*b*) the solutions are given by (5.6) and in principle one has now the possibility of unstable solutions due to exponential growth. The Floquet solutions can be obtained making $A \neq 0$ and $B = 0$ or vice versa. Only the first case is dangerous since the solution can grow and one has

$$z = \exp \left(\frac{1}{\Omega} \int_0^{2\pi} \mathrm{d}\varphi (E - V(\varphi))^{1/2} \right), \tag{5.9}$$

$$\mu = -\bar{\gamma}_k + \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi (E - V(\varphi))^{1/2}. \tag{5.10}$$

Making the change of variables $u = \cos\varphi$ we obtain

$$\mu = -\bar{\gamma}_k \frac{V_0^{1/2}}{\pi} \int_{-1}^{1} \mathrm{d}u \left(\frac{u - E/V_0}{1-u^2} \right)^{1/2} \tag{5.11}$$

and using this last expression one can show that $\mu < 0$ always and this case is then also stable. The property $\mu < 0$ can be shown by noticing that due to $E < -V_0 < 0$ one can considerer the exponent as a function $\mu(\Gamma_k)$ of Γ_k with

$$\mu(0) = (\bar{\gamma}_k^2 - \bar{\omega}_k^2)^{1/2} - \bar{\gamma}_k < 0$$

and such that $\mathrm{d}\mu/\mathrm{d}\Gamma_k < 0$.

The sector $|E| < V_0$ (see figure 6*c*) is the most interesting one since it is in this region that we have instability. The solutions are now a combination of the functions (5.5) and (5.6) which complicates the problem since one has to connect solutions between regions with $E - V(x) > 0$ and regions with $E - V(x) < 0$ and vice versa. This is done using the WKB connection formulas (Messiah 1973) and after the construction of the Floquet solutions which are now linear combinations with A, $B \neq 0$, one can show that the factor z satisfies the quadratic equation (Goldman & Krivchenkov 1961)

$$z^2 - 2X\,z + 1 = 0, \tag{5.12}$$

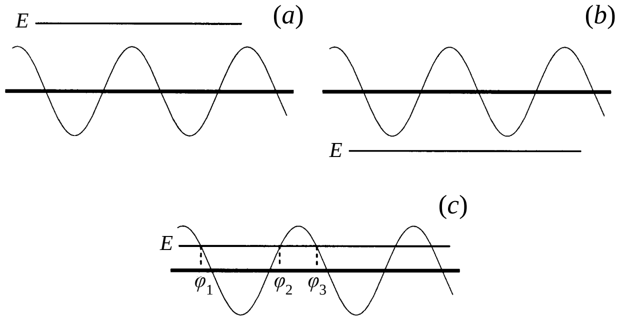

FIGURE 6. Representation of the energy E of a quantum particle with respect to the potential $V(x)$. (*a*) $E > V_0$, (*b*) $E < -V_0$, (*c*) $|E| < V_0$.

with

$$X = \cos p\,(e^{q} + \tfrac{1}{4}e^{-q}), \tag{5.13}$$

$$p = \frac{1}{\Omega}\int_{\varphi_1}^{\varphi_2} d\varphi (E - V(\varphi))^{1/2}, \quad q = \frac{1}{\Omega}\int_{\varphi_2}^{\varphi_3} d\varphi (V(\varphi) - E)^{1/2}, \tag{5.14}$$

where the points φ_1, φ_2 and $\varphi_3 = \varphi_1 + 2\pi$ are defined in figure 6(*c*) and correspond to classical turning points. The roots z_+ and z_- of (5.12) determine the Floquet exponents and since $z_+z_- = 1$ it is sufficient to know one of them to find the second one (this is a general property of equation (5.1), see for example Landau & Lifshitz 1970). The exponent which can lead to instability is

$$\mu = \begin{cases} -\bar{\gamma}_k + i(\Omega/2\pi)\operatorname{arccos} X, & |X| < 1 \\ -\bar{\gamma}_k(\Omega/2\pi)\operatorname{arccosh} X, & X > 1 \\ -\bar{\gamma}_k(\Omega/2\pi)\operatorname{arccosh}|X| + i(\Omega/2), & X < -1. \end{cases} \tag{5.15}$$

In the first case $|X| < 1$ the amplitude changes its phase and is damped and is then always stable. In the other two cases the amplitude may grow. When $X < -1$ the amplitude changes its phase by π in one period which corresponds to a subharmonic solution and when $X > 1$ the phase does not change in a period and we have an harmonic solution. From these solutions one can understand the appearance of the tongues since the value of X is a rapidly oscillating function of the wavenumber due to the presence of the factor $\cos p$ and the solution will change from harmonic to oscillating to subharmonic and vice versa when the wavenumber is varying and then the space (Γ, k) will be necessarily divided into sectors which represent the tongues.

5.2. *A new condition of resonance*

In the weakly viscous case the resonant modes satisfy approximately the relation $\omega_k \approx n\Omega/2$, and we have now a different situation. For a well-defined threshold Γ we can determine the modes which are more likely to be excited. Due to the value of the Floquet exponent these modes will be those which satisfy $\cos p = \pm 1$ (since they deviate more from $|X| < 1$ which is stable) with the positive sign corresponding to an harmonic solution and the negative sign to a subharmonic one. The condition is then $p = n\pi$ and after a change of variables (5.14) can be expressed as

$$\frac{V_0^{1/2}}{\pi}\int_{-E/V_0}^{1} du \left(\frac{u + E/V_0}{1 - u^2}\right)^{1/2} = n\frac{\Omega}{2}, \tag{5.16}$$

which is the new resonance condition. Since the left-hand side of (5.16) is always positive $n = 1, 2, 3....$, and $n = 1$ corresponds to the first subharmonic tongue, $n = 2$ to the second harmonic, etc. Since the integral is always of $O(1)$ the condition is approximately of the form

$$\frac{\bar{\omega}_k \Gamma_k^{1/2}}{\pi} \sim n\frac{\Omega}{2}, \tag{5.17}$$

which is similar to the case of a weakly damped pendulum except for the extra factor $\Gamma_k^{1/2}$ which changes strongly the resonance condition since the threshold Γ for the instability of a mode depends strongly on the mode. For fixed Γ the term on the left of equation (5.17) increases with the wavenumber and then the tongues (which are characterized by each one of these resonant modes) have a characteristic mode which increases with n.

5.3. *The instability threshold of each mode k*

A resonant mode is not necessarily unstable and we still have to find the minimum threshold for the instability of each tongue. The instability of the subharmonic and harmonic solutions depends on the value of the real part of the Floquet exponent

$$\text{Re}\,(\mu) = \frac{\Omega}{2\pi}\text{arccosh}\,|X| - \bar{\gamma}_k. \tag{5.18}$$

We are interested in the stability of the resonant modes, i.e. $\cos p = \pm 1$, and since the WKB approximation has strict validity when $\ell \ll \delta$ we have $q \gg 1$ (from its definition we can see that $q \approx \delta^2/\ell^2$) and the value of X can be approximated by $X \approx \pm \text{e}^q$. Then $\text{arccosh}\,|X| \approx q + \ln 2 \approx q$ and we obtain for the real part of the exponent

$$\text{Re}\,(\mu) = \frac{\Omega}{2\pi}q - \bar{\gamma}_k. \tag{5.19}$$

We replace q and after a change of variable we obtain

$$\text{Re}\,(\mu) = \frac{V_0^{1/2}}{\pi}\int_{E/V_0}^{1} \text{d}u \left(\frac{u - E/V_0}{1-u^2}\right)^{1/2} - \bar{\gamma}_k. \tag{5.20}$$

When $|E| < V_0$ the integral is of $O(1)$ and one can estimate $\text{Re}\,(\mu) \approx V_0^{1/2}/\pi - \bar{\gamma}_k$. The exponent is then the result of a competition between the amplifying effect represented by $V_0^{1/2}$ and the dissipative effect represented by $\bar{\gamma}_k$.

In order to study the dependence on the wavenumber k of the Floquet exponent we observe that $V_0^{1/2} = \Gamma^{1/2}\bar{\omega}_k/(1 + \tau k^2/\rho g)^{1/2}$ and then a way to compare amplification and dissipation when the wavenumber is varying is to study the quantity $(\bar{\gamma}_k/\bar{\omega}_k)^2(1 + \tau k^2/\rho g)$ which is represented in figure 5. We can see in that figure that when k is big the dissipative effect dominates and those modes are very stable, i.e. they will have a high threshold, while for small k the amplification is less effective in front of a dissipation which saturates ($\bar{\gamma}_k(kh \ll 1) \approx 5\nu/4h^2$) and we have again a high threshold. This explains the appearance of an intermediate mode k_* which makes the amplification more effective and the dissipation smaller: this mode is then the most unstable one if it is resonant; it is not precisely the minimum of figure 5 but is related to it. As was shown in § 4, the existence of this mode k_* allows one to understand the existence of a series of bicritical points (see Kumar 1996) where a new resonant mode is the most unstable.

We study now the curve of marginal stability Re $(\mu) = 0$. Returning to the original parameters in the Mathieu equation this curve is given by

$$\frac{1}{\pi}\int_{\frac{1-(\bar{\gamma}_k/\bar{\omega}_k)^2}{\Gamma_k}}^{1} \mathrm{d}u \left(\frac{\Gamma_k u + (\bar{\gamma}_k/\bar{\omega}_k)^2 - 1}{1-u^2}\right)^{1/2} = \frac{\bar{\gamma}_k}{\bar{\omega}_k}. \tag{5.21}$$

This expression can be interpreted as a relation between the parameters Γ_k and $\bar{\gamma}_k/\bar{\omega}_k$ which defines implicitly a dimensionless function $\Delta(z)$ through

$$\Gamma_k = \Delta(\bar{\gamma}_k/\bar{\omega}_k). \tag{5.22}$$

This function $\Delta(z)$ is very simple and has the following properties which follow from its definition (see Appendix B): $\Delta(0) = 1$, $\Delta'(0) = 2\sqrt{2}$, $\Delta' > 0$, $\Delta''(0) > 0$ and $\Delta(z > 1) \approx z^2/a^2 + b$, with $a \approx 0.481$ and $b \approx 3.21$. Comparing equations (4.3) and (5.22), we observe that the asymptotic form of the function $\Delta(z)$ makes these results equivalent.

We can discuss now the stability in the space (Γ, k). Replacing the value of Γ_k, the marginal stability curve can be written as

$$\Gamma = \Delta(\bar{\gamma}_k/\bar{\omega}_k)(1 + \tau k^2/\rho g). \tag{5.23}$$

The curve has a minimum which is determined exclusively through the dependence of Γ on $\bar{\gamma}_k/\bar{\omega}_k$ in (5.23) since the function Δ is always increasing $(\Delta' > 0)$. This minimum defines a point (Γ_*, k_*) which had been detected naively in § 4, but an exact calculation requires using equation (5.23).

5.4. *Behaviour of the resonant modes on the frequency and existence of a frequency cutoff*

Once the threshold of each mode k is determined it is possible to study through the resonance condition (5.16) the most unstable mode in each tongue. Replacing the value of Γ given by (5.23) in the expression (5.16) we obtain

$$\frac{\bar{\omega}_k \Delta(u_k)^{1/2}}{\pi}\int_{-(1-u_k^2)/\Delta(u_k)}^{1} \mathrm{d}u \left(\frac{(1-u_k^2)/\Delta(u_k) + u}{1-u^2}\right)^{1/2} = n\frac{\Omega}{2}, \tag{5.24}$$

where $u_k = \bar{\gamma}_k/\bar{\omega}_k$. Equation (5.24) gives the wavenumber as a function of the frequency and of the value of n. In order to get some insight the easiest thing to do is to determine the dependence in limit cases. When the wavenumber k is big one has $u_k \gg 1$ and we can use the asymptotic form $\Delta(z) \approx z^2/a^2 + b \approx z^2/a^2$; then equation (5.24) can be approximated by

$$\frac{\bar{\gamma}_k}{a\pi}\int_{a^2}^{1} \mathrm{d}u \left(\frac{-a^2 + u}{1-u^2}\right)^{1/2} = n\frac{\Omega}{2} \tag{5.25}$$

and since $\bar{\gamma}_k \approx 2\nu k^2/3$ for big k one has

$$k = \frac{c\, n^{1/2}}{\delta}, \tag{5.26}$$

where $\delta = (\nu/\Omega)^{1/2}$ and the constant c has the value $c \approx 1.12$. This result is surprising if one thinks that for weak viscosity the wavenumber is strongly determined by the surface tension at high frequency. At high frequency, the resonance with a minimum threshold is the first subharmonic $n = 1$, therefore the most unstable mode is given by $k_c = c/\delta$.

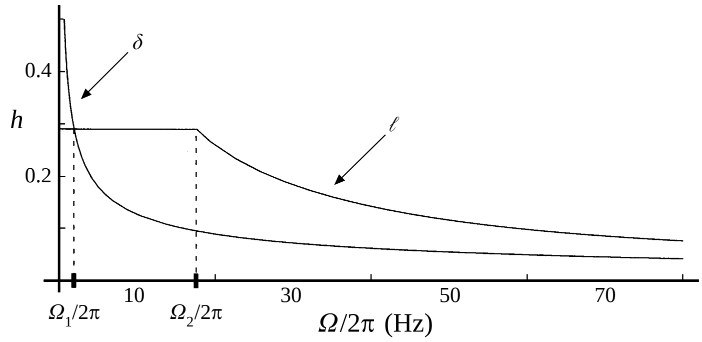

FIGURE 7. Variation of ℓ and δ with frequency for the parameters of figure 8.

When the wavenumber is small we have again that $u_k \gg 1$ and we can proceed in a similar way. Instead of equation (5.25) we obtain

$$\frac{\bar{\gamma}_k}{a\pi}\int_{a^2}^{1} \mathrm{d}u \left(\frac{-a^2+u}{1-u^2}\right)^{1/2} = n\frac{\Omega}{2}, \tag{5.27}$$

but now the dissipation $\bar{\gamma}_k$ saturates for small k and takes the value $\bar{\gamma}_k \approx 5\nu/4h^2$ which does not depend on the wavenumber. This occurs for a well defined value of the frequency which is given by (5.27)

$$\frac{\Omega_{cf}^{(n)}}{2\pi} = \frac{d}{n}\frac{\nu}{h^2}, \tag{5.28}$$

with $d = 5/(8c^2\pi) \approx 0.238$. We observe that the $\Omega_{cf}^{(n)}/2\pi$ are frequencies cutoff: at frequencies smaller than $\Omega_{cf}^{(n)}/2\pi$ no mode in the nth tongue can be excited. Another interesting point is that by consistency we find a lower bound on the frequencies for the appearance of the first bicritical point which is that it can only be present for frequencies greater than $\Omega_{cf}^{(1)}/2\pi$.

5.5. *Conditions for the validity of the lubrication approximation*

We are able now to obtain the conditions on the experimental parameters such as frequency, viscosity, size of the fluid layer, etc. to satisfy the restriction $\ell \lesssim \delta$, which is neccesary to use the lubrication approximation.

There is a situation in which one has a sufficient condition to use the lubrication approximation: since $\ell < h$ it is enough to ask that $h \lesssim \delta$ and this condition can be written as

$$\Omega \lesssim \Omega_1 = \frac{\nu}{h^2}, \tag{5.29}$$

where the right-hand side defines a characteristic frequency and in what follows we refer to high or low frequency in the sense of comparing the frequency of the forcing with Ω_1.

On the other hand, we know that at high frequency the most unstable mode scales as $k_c \sim 1/\delta$, therefore the length ℓ is given by $\ell \sim \delta$. Consequently, we are able to use the lubrication approximation again.

There is difficulty in verifying the condition $\ell \lesssim \delta$ for intermediate frequencies. Using the definition (2.23) and the resonance condition (5.17), we can compute the value of ℓ and compare it with the value of δ as shown in figure 7. The region in which there is a large deviation from the condition $\ell \lesssim \delta$ is near the frequency Ω_2

(where $\ell = h$), and if we can have conditions such that $\Omega_2 \approx \Omega_1$ we can be assured of the validity of the lubrication approximation for the whole spectrum of frequencies.

The frequency Ω_2 is given by the resonance condition

$$\frac{\Omega_2}{2} \approx \left. \frac{\bar{\omega}_k \Gamma_k^{1/2}}{\pi} \right|_{k=1/h}. \tag{5.30}$$

Using the asymptotic approximation $\Delta(z) \approx z^2/a^2 + b$ we find

$$\frac{\Omega_2}{2} \approx \frac{1}{\pi} \left. (\bar{\gamma}_k^2/a^2 + b\,\bar{\omega}_k^2)^{1/2} \right|_{k=1/h} \tag{5.31}$$

$$= \frac{\Omega_1}{2\pi} \left(\left(\frac{C_1(1)}{aC_2(1)} \right)^2 + 4\,b \frac{\tanh(1)}{C_2(1)} \frac{gh^3}{\nu^2} (1 + \tau/\rho h^2 g) \right)^{1/2}. \tag{5.32}$$

If we want both frequencies Ω_1 and Ω_2 to be of the same order the term inside the square root should have a value with a very weak dependence on the rest of the parameters. This can only happen if

$$e \frac{\alpha}{1+\beta} \gtrsim 1, \tag{5.33}$$

where the dimensionless parameters are $\alpha = \nu^2/h^3 g$, $\beta = \tau/\rho h^2 g$ and e has the value

$$e = \frac{C_2(1)}{b \tanh(1)} \left(\frac{C_1(1)}{2aC_2(1)} \right)^2 \approx 16.6. \tag{5.34}$$

Relation (5.33) is the condition we wanted. To check this relation, we use the experimental results which were given in Kumar (1996) and Kumar & Tuckerman (1996) (these experimental data were obtained by W. S. Edwards) for the experimental parameters: $\nu = 1.02$ cm s^{-2}, $\rho = 1.2$ g cm^{-3}, $\tau = 67.6$ erg cm^{-2} and $h = 0.29$ cm. The term on the left of (5.33) has the value 0.4 for these parameters. Although this value is small, we observe a good agreement between the experimental results and the simulation of the CT equation as we show in figure 8.

5.6. *Study of the most unstable mode k_* and of the minimal threshold Γ_**

When the frequency is low, i.e. it is of $O(\nu/h^2)$ or smaller, the most unstable mode and the minimal threshold which are the observable quantities which characterize the instability are determined by k_* and Γ_*. From their definition these quantities do not depend on the frequency which is a remarkable physical property which also simplifies the analysis. We want to study how k_* and Γ_* depend on the parameters of the fluid. The curve of marginal stability (5.23) can be written in terms of $y = kh$ and of two dimensionless parameters $\alpha = \nu^2/h^3 g$ and $\beta = \tau/\rho h^2 g$ which we have already found in the condition for the application of the lubrication approximation. From (5.23) we obtain

$$\Gamma = \Delta \left(\frac{\alpha^{1/2}}{(1+\beta y^2)^{1/2}} \chi(y) \right) (1 + \beta y^2), \tag{5.35}$$

$$\chi(y) = \frac{y^{3/2}}{2} \frac{C_1(y)}{(C_2(y) \tanh y)^{1/2}}. \tag{5.36}$$

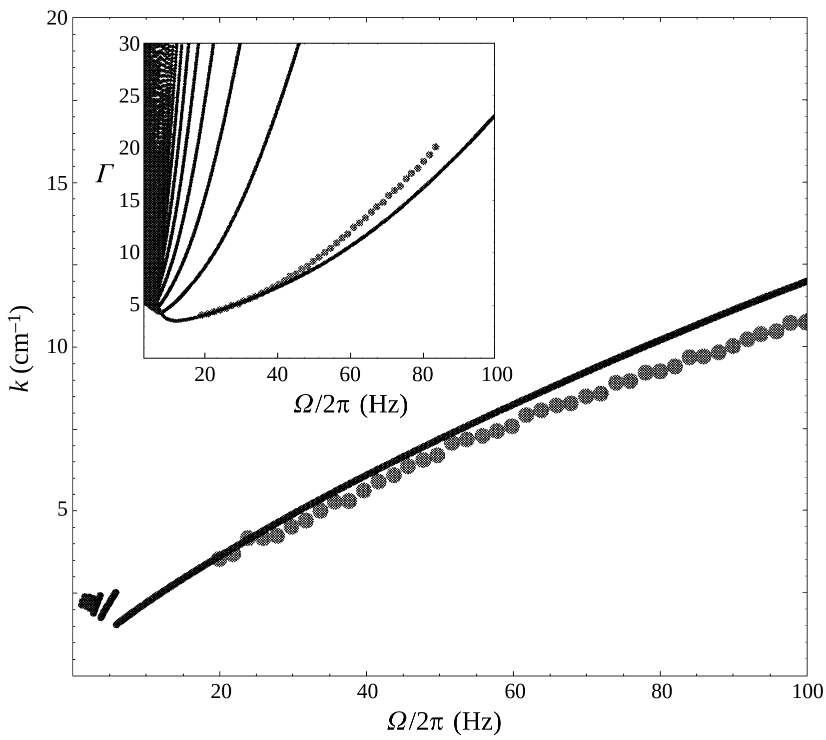

FIGURE 8. Numerical simulation of the CT equation compared with experimental data: variation of wavenumber with frequency. Black circles correspond to experiments and lines to the simulation. In the inset: variation of minimal threshold with frequency. The lines which fit the data correspond to the first subharmonic resonance.

A complete study can be done about the minimum (Γ_*, y_*) of this curve. Both coordinates will be functions of the two parameters α and β and we shall write

$$\Gamma_* = \Gamma_*(\alpha, \beta), \quad k_* = \frac{y_*(\alpha, \beta)}{h}. \tag{5.37}$$

The analysis is simpler if we consider a situation in which we can neglect the surface tension ($\beta = 0$); then the curve of marginal stability is

$$\Gamma = \Delta(\alpha^{1/2}\chi(y)). \tag{5.38}$$

Since the function $\Delta(z)$ is always increasing the minimum of this curve is given by the minimum (χ_m, y_m) of the function $\chi(y)$ which does not depend on the parameter α. A numerical calculation gives $y_m \approx 0.859$ and $\chi_m \approx 3.48$. The most unstable mode and the minimal threshold are then given by

$$\Gamma_* = \Delta(\alpha^{1/2}\chi_m), \quad k_* = \frac{y_m}{h}. \tag{5.39}$$

The most unstable mode is determined by the height h of the fluid layer. On the other hand the minimal threshold Γ_* is determined by the value of the parameter α and this can be interpreted as a relation between dissipation and injection of energy since α depends on the viscosity and the height of the fluid layer which are the quantities that determine the dissipation and on the other hand Γ is related to the injection of

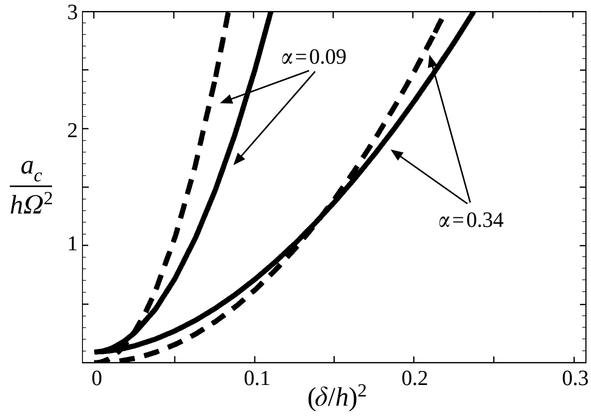

FIGURE 9. Plots of scaling (5.40) (dashed line) and scaling (5.41) (solid line) for the limit values $\alpha = 0.09$ and $\alpha = 0.34$ used by Lioubashevsky *et al.* Notice that the α parameter represents the degree of dissipation in the fluid, and the agreement between both scalings is better when α is larger.

energy in the system. Notice that if the asymptotic form of $\Delta(z)$ is used the threshold Γ_* can be easily calculated with the result $\Gamma_* \approx 52.4\alpha + 3.21$.

We can compare this result with the experimental results of Lioubashevsky *et al.* (1997) who have found through a numerical fitting of the experimental data a relation between the threshold of acceleration, a_c, and the driving frequency Ω. They identified experimentally three relevant dimensionless parameters: $a_c/(h\Omega^2)$, $\nu/(\Omega h^2) \equiv (\delta/h)^2$ and $g/(h\Omega^2)$ which obey the scaling behaviour $(2a_c/\pi - g)/(h\Omega^2) = 0.059 + 21.4(\delta/h)^{3.54}$. Using the CT equation we find only two dimensionless parameters as explained above. However there is no contradiction with the scaling of Lioubashevsky *et al.* (1997) since we can show that both scalings are equivalent.

In order to compare these 'different scalings', we choose the dimensionless parameters $a_c/(h\Omega^2)$, $(\delta/h)^2$ and α. Lioubashevsky *et al.*'s scaling and our scaling are

$$a_c/(h\Omega^2) = \pi/2\left(0.059 + 21.4(\delta/h)^{3.54} + (\delta/h)^4/\alpha\right), \tag{5.40}$$

$$a_c/(h\Omega^2) = 52.4(\delta/h)^4 + 3.21(\delta/h)^4/\alpha, \tag{5.41}$$

respectively. A plot of these relations is presented in figure 9 where we compare both scalings for the minimum and maximum experimental values of the α parameter chosen in Lioubashevsky *et al.* (1997).

The general case which considers surface tension can be analysed as follow. We can show rigorously using (5.36) that for all values of the parameters one has $y_*(\alpha, \beta) \leqslant y_m$. This indicates that the wavelength of the structure $2\pi/k_* = 2\pi h/y_*$ is always bigger than h and in relation with our previous result for $\beta = 0$ this length is not too sensitive to variations of the parameters α and β. More precisely one has $y_*(\alpha > \beta, \beta) \approx y_m$. This can be seen looking for the conditions needed to recover the results obtained neglecting the surface tension. Using again the asymptotic form $\Delta(z) \approx z^2/a^2 + b$ in (5.36) we obtain

$$\Gamma \approx \alpha\chi(y)^2/a^2 + b(1 + \beta y^2) \tag{5.42}$$

and since we want the marginal curve to be similar to the one obtained for $\beta = 0$ near the point (χ_m, y_m), i.e. near the minimum for $\beta = 0$, we require

$$f\frac{\alpha}{\beta} \gg 1, \tag{5.43}$$

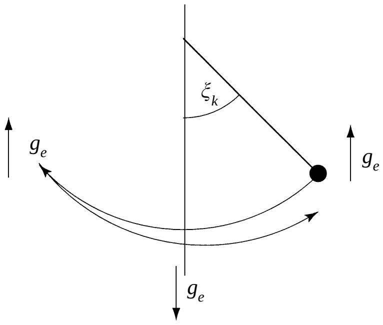

FIGURE 10. The subharmonic case with $n = 1$ corresponds to amplification at half the period, just at the moment the effective gravity $g_e(t)$ points upward.

with $f = \chi_m^2/(a^2 b\, y_m^2) \approx 88.6$. For instance, the term on the left of (5.43) is approximately 5.6 for values of figure 8; then in this case the surface tension can be neglected in the low-frequency regime and we can use the previous analysis.

5.7. *The origin of the instability in the lubrication approximation*

The minimal threshold for instability is $\Gamma_k \geqslant 1$ according to (5.22). This means that the system spends intervals of time subjected to effective negative accelerations and it is precisely in these intervals where the amplification occurs. This can be seen as follows: the WKB solutions (5.5) and (5.6) show that amplification can only occur when $E - V(x) < 0$ (in fact one must have $|E| < V_0$ since $E < -V_0$ is always stable) and the best time interval in each period to satisfy the condition is near a maximum of $V(x)$ where the acceleration of the plate is negative ($V(x) \sim -\ddot{z}_p$). Subjecting the fluid to negative accelerations is the origin of the instability. Physically this situation corresponds to putting a heavier fluid over a lighter one (air) as in the Rayleigh–Taylor instability (Chandrasekar 1981) which is immediately unstable (it is enough to have different densities). In our case this changes since the threshold may be of several 'negatives g' because the fluid is not subjected to negative accelerations all the time and when it is not one can see in (5.5) that the surface oscillates and is damped with a frequency given approximately by $\omega_0 \approx \bar{\omega}_k \Gamma_k^{1/2}/\pi$. The effect in this second step is then in opposition to the amplification due to dissipation and this leads to a higher instability threshold. These considerations also explain the existence of a best time for amplification of the system: there should be amplification (negative accelerations) when the amplitude of the surface is greatest since in this way the potential energy that already exists is being amplified. It is simpler to see this with the analogy of a dissipative pendulum whose rotation axis is oscillating. The pendulum which obeys equation (3.15) is subjected to negative gravity at time intervals of $2\pi/\Omega$ and it is best to amplify it whenever it has reached its greatest amplitude (see figure 10), i.e. at an integer multiple of half its 'effective period'

$$\frac{n}{2}\left(\frac{2\pi}{\omega_0}\right) \approx \frac{2\pi}{\Omega}, \tag{5.44}$$

which is the resonance condition (5.17).

6. Conclusions

We enumerate the most important results we have obtained using our general equation (2.17) and the CT equation (3.15):

(i) The dimensionless parameter involved in the different regimes of Faraday's phenomenon is δ/ℓ. In addition to the usual interpretation in terms of comparing two lengths, δ which characterizes the size of the boundary layer and ℓ the penetration length of the fluid motion, we have given another interpretation to the fact that $\delta = (\nu/\Omega)^{1/2}$ can have any value which can even be bigger than the size of the fluid layer h (when the viscosity is sufficiently big or the frequency sufficiently small). We have shown that the quantity δ/ℓ represents the length of the memory in the non-local equation (2.17): if $\delta/\ell \ll 1$ the memory time is long and vice versa.

(ii) It was well-known that in the case $\delta \ll \ell$ a Mathieu equation could describe the system. We have shown that when $\delta \gtrsim \ell$ the system is described by the Cerda–Tirapegui equation which is also of the Mathieu type but with different coefficients. It is a systematic approximation obtained from equation (2.17) in powers of a small parameter $\Omega\tau_d$ and so is much more controllable than the Mathieu equation for weak viscosity. We remark that even if a fluid is weakly viscous if one forces at sufficiently high frequencies one will excite very short wavelengths for which dissipation is very efficient and one will necessarily be in a regime of strong dissipation and then in the lubrication approximation. The reason is that the condition to have a weakly viscous fluid is $k\delta \ll 1$ (in deep water) where the mode k is given by the resonance condition $\omega_k = \Omega/2$; at high frequency this relation tells us that the mode k increases with frequency faster than $\Omega^{1/2}$ and it is then impossible to satisfy $k\delta \ll 1$. At frequencies where the previous situation occurs the results of the lubrication approximation can be used and they give a wavelength which varies as $k^{-1} \sim (\nu/\Omega)^{1/2}$ according to equation (5.26). This has been observed by S. Fauve *et al.* (1992) studying Faraday's instability for a fluid near the critical point. When the frequencies are sufficiently low, smaller than ν/h^2, one is again in the lubrication approximation. We can see then that (paradoxically!) it is easier to give conditions to be always in the lubrication regime where the CT equation is valid than it is to be in the weakly viscous regime. We estimate that the relevant condition is (5.33).

(iii) The analytical study of the Mathieu equation is difficult in itself. But due to the fact that the dissipative term in the CT equation is big enough it is possible to use the WKB approximation. This method is accurate when the parameter ℓ/δ is big and leads then to analytical results which are quantitatively correct when $\delta \gg \ell$ and qualitative when $\delta \approx \ell$. The WKB technique is a perturbative method which gives more global information than the methods which are used in the weakly dissipative case (Landau & Lifshitz 1970; Bergé *et al.* 1984) and this advantage allows one to explore analytically in a rather simple way the CT equation and hence to understand how the instability occurs in the strongly viscous case.

(iv) Thanks to our analytical results we can show that for strong viscosity the forcing alone can provoke the inertial behaviour. This can be see in the existence of an effective frequency $\approx \bar{\omega}_k \Gamma_k^{1/2}$ at which the system oscillates. On the other hand one can recognize that the amplification is due essentially to the fact that the system spends time in regions of effective negative accelerations which is an unstable situation as the Rayleigh–Taylor instability shows and due to this the amplification is much more evident than in the weakly viscous case. The amplification is parametric because the system organizes itself in such a way as to synchronize the effective frequency of the forcing so as to allow for the amplification of the amplitude of the surface at the best moment, and this fact is the origin of a resonance condition as in the weakly viscous case and of the appearance of the behaviours associated with each resonance (harmonic and subharmonic behaviours).

(v) Studying the threshold of each tongue we can understand analytically why at a frequency of $O(\nu/h^2)$ the instability ceases to be subharmonic and why the system goes through a series of bicritical points if one continues lowering the frequency. The essential reason for this is that the ratio between dissipation and amplification $(\bar{\gamma}_k/\bar{\omega}_k)$ is minimized as a function of the wavenumber at a non-vanishing value of k.

(vi) Finally we have analytic access to the most unstable wavelength and threshold for the forcing which are two quantities of direct experimental interest. At low frequencies we have found two relevant dimensionless parameters which determine the threshold and the wavelength: they are $\alpha = \nu^2/h^3 g$ and $\beta = \tau/\rho h^2 g$. If the value of β can be neglected (we give conditions for this in equation (5.43)) we conclude that the threshold is proportional to α and that the wavelength is proportional to h, the size of the fluid layer.

We thank Francisco Melo (U.S.A.CH.), Pierre Collet (Ecole Polytechnique) and L. Mahadevan (M.I.T.) for valuable discussions on this topic. We acknowledge support from projects Fondecyt, C.E.E., E.C.O.S. and Cátedra Presidencial en Ciencias 1997.

Appendix A. Solution of the diffusive problem

A.1. *The problem to be solved*

In order to calculate the diffusive velocity $u_{z\boldsymbol{k}}$ as a functional of the unknown amplitude of the surface $\xi_{\boldsymbol{k}}$ we must solve

$$[\partial_t - \nu(\partial_z^2 - k^2)]u_{z\boldsymbol{k}}(z,t) = 0, \tag{A 1}$$

$$u_{z\boldsymbol{k}}(z,t)|_{z=0} = -2\nu k^2 \xi_{\boldsymbol{k}}(t), \tag{A 2}$$

$$[\sinh kh\, \partial_z u_{z\boldsymbol{k}}(z,t) + k \cosh kh\, u_{z\boldsymbol{k}}(z,t)]|_{z=-h} = -k(\partial_t + 2\nu k^2)\xi_{\boldsymbol{k}}(t). \tag{A 3}$$

These equations are not sufficient to determine the problem. Analogous problems are those of thermal diffusion where our boundary conditions would be walls with a temperature varying in time. It is necessary to give the following conditions to specify completely the solution (Morse & Feshbach 1953):

$$u_{z\boldsymbol{k}}(z,t)|_{t=t_0} = u_{z\boldsymbol{k}}(z,t_0), \quad \text{and causality.} \tag{A 4}$$

The first one corresponds to an initial condition and due to the self-consistency of the equations it is not possible to take any velocity profile since it must necessarily satisfy the boundary conditions (A 2)–(A 3) and be related to a given amplitude of the surface at the initial time $t = t_0$. With respect to causality we must eliminate solutions which depend on the 'temperature' of the boundaries at future times. The equation can be simplified with the transformation (see Beyer & Friedrich 1995)

$$u_{z\boldsymbol{k}}(z,t) = \phi(z,t)\mathrm{e}^{-\nu k^2(t-t_0)},$$
$$\xi_{\boldsymbol{k}}(t) = \eta(t)\mathrm{e}^{-\nu k^2(t-t_0)},$$

which gives from (A 1)–(A 3)

$$(\partial_t - \nu\partial_z^2)\phi(z,t) = 0, \tag{A 5}$$

$$\phi(z,t)|_{z=0} = -2\nu k^2 \eta(t), \tag{A 6}$$

$$[\sinh kh\,\partial_z \phi(z,t) + k\cosh kh\,\phi(z,t)]|_{z=-h} = -k(\partial_t + \nu k^2)\eta(t), \tag{A 7}$$

and we also write the initial condition as $\phi(z,t)|_{t=t_0} = \phi_0(z)$.

A.2. *Solution by Laplace transform*

The Laplace transform defined by

$$\phi_s(z) = \int_{t_0}^{\infty} \mathrm{d}t\,\mathrm{e}^{-s(t-t_0)}\phi(z,t)$$

transforms equation (A 5) to

$$(s - \nu\partial_z^2)\phi_s(z) = \phi_0(z), \tag{A 8}$$

which incorporates the initial condition. We define

$$f_1(t) = -2\nu k^2 \eta(t),$$
$$f_2(t) = -k(\partial_t + \nu k^2)\eta(t).$$

Then equations (A 6)–(A 7) give

$$\phi_s(z)|_{z=0} = \tilde{f}_1, \tag{A 9}$$

$$[\sinh kh\,\partial_z \phi_s(t) + k\cosh kh\,\phi_s(z)]|_{z=-h} = \tilde{f}_2, \tag{A 10}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are the Laplace transforms of f_1 and f_2, respectively.

The solution of the homogeneous equation $(s - \nu\partial_z^2)\phi_s(z) = 0$ is

$$\phi_s^{(h)}(z) = A_s \sinh q(z+h) + B_s \cosh q(z+h), \tag{A 11}$$

where $q = (s/\nu)^{1/2}$. The constants A_s and B_s will be determined by the boundary conditions. We need now a particular solution $\phi_s^{(p)}$ of (A 8) which we take as

$$\phi_s^{(p)}(z) = \frac{1}{\nu q}\int_{-h}^{z} \mathrm{d}z' \sinh q(z'-z)\phi_0(z'). \tag{A 12}$$

The general solution of (A 8) can be written

$$\phi_s = \phi_s^{(h)} + \phi_s^{(p)} \tag{A 13}$$

and now we have to apply our boundary conditions to it. Using (A 9)–(A 10) we determine A_s and B_s and then the homogeneous solution can be written

$$\phi_s^{(h)}(z) = (\tilde{f}_1 - \phi_s^{(p)}(0))\tilde{K}_1(s,z) + \tilde{f}_2\tilde{K}_2(s,z). \tag{A 14}$$

The functions $\tilde{K}_1$ and $\tilde{K}_2$ are

$$\tilde{K}_1(s,z) = \left(\frac{k\sinh q(z+h)\cosh kh - q\cosh q(z+h)\sinh kh}{k\sinh qh\cosh kh - q\sinh kh\cosh qh}\right), \tag{A 15}$$

$$\tilde{K}_2(s,z) = \left(\frac{\cosh q(z+h)\sinh qh - \sinh q(z+h)\cosh qh}{k\sinh qh\cosh kh - q\sinh kh\cosh qh}\right). \tag{A 16}$$

These two functions have simple poles due to zeros in the denominator. An analysis of the denominator shows a zero at $q = k$ which in the complex s-plane corresponds to $s_* = \nu k^2$ (this zero will finally give no contribution) and a zero at $q = 0$ but this is not a pole since the numerator also vanishes. The other zeros form an infinite set on

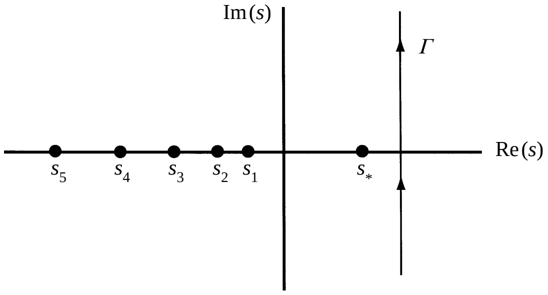

FIGURE 11. Singularities in the complex plane of the functions $\tilde{K}_1$ and $\tilde{K}_2$ and the appropriate path Γ which ensures causality.

the negative real axis. The set is determined by looking for zeros of the form $q = \mathrm{i}p$ with p real; then one has

$$\frac{\tan ph}{ph} = \frac{\tanh kh}{kh} \tag{A 17}$$

and this equation has a numerable set of solutions for p which can be written in increasing order as $\{p_m\}_{m:1..\infty}$. In the complex s-plane the zeros have the form $s_m = -\nu p_m^2$ and they move away from the origin with increasing m. All this is summarized in figure 11. For example in the case of deep water ($kh \gg 1$) one has approximately $p_m \approx m\pi/h$ and then $s_m \approx -\nu m^2\pi^2/h^2$ which shows that the distance between the zeros and between any zero and the origin decreases when $h \to \infty$ and increases in the opposite case.

A.3. *Solutions in real space*

In order to come back to real space we have to make an inverse Laplace transform of the solution (A 13)

$$\phi(z,t) = \int_\Gamma \frac{\mathrm{d}s}{2\pi\mathrm{i}}\, \mathrm{e}^{s(t-t_0)}\phi_s(z), \tag{A 18}$$

where the integral is done in the complex s-plane on a path Γ which one must specify. Using the explicit form of ϕ_s we obtain

$$\phi(z,t) = \int_{t_0}^{\infty} \mathrm{d}t' f_1(t')K_1(t-t',z) + \int_{t_0}^{\infty} \mathrm{d}t' f_2(t')K_2(t-t',z) + \int_{-h}^{0} \mathrm{d}t'\phi_0(z')K_3(t-t_0,z',z), \tag{A 19}$$

where the kernels K_1 and K_2 are the inverse transforms of $\tilde{K}_1$ and $\tilde{K}_2$, respectively, and K_3 was defined in an analogous way. The path Γ in (A 18) must be to the right of all singularities of the integrand in the complex s-plane and this ensures causality. For example for K_1 one has

$$K_1(\sigma,z) = \int_\Gamma \frac{\mathrm{d}s}{2\pi\mathrm{i}}\, \mathrm{e}^{s\sigma} \left(\frac{k \sinh q(z+h)\cosh kh - q\cosh q(z+h)\sinh kh}{k\sinh qh\cosh kh - q\sinh kh\cosh qh} \right) \tag{A 20}$$

and with Γ as stated above one has

$$K_1(\sigma,z) = 0 \quad \text{if } \sigma < 0.$$

In figure 11 we have drawn Γ and we can see that for $\sigma < 0$ the path of integration can be closed with a big semi-circle of infinite radius on the right which gives zero for the integral since one has no singularities inside the closed path. One also can check that the contribution of the pole at s_* of the kernels cancels in (A 19) as it must since it would give in real space an unphysical contribution of the form const. $\times\, e^{s_* \sigma}$. The final result can be written

$$\phi(z,t) = \int_{t_0}^{\infty} dt' \eta(t') K(t-t',z) + \int_{-h}^{0} dz' \phi_0(z') G(t-t_0,z',z), \tag{A 21}$$

with the new kernels given by

$$K(\sigma,z) = \int_{\Gamma} \frac{ds}{2\pi i}\, e^{s\sigma} \tilde{K}(s,z),$$

$$\tilde{K}(s,z) = -2\nu k^2 \cosh qz + \nu k \sinh qz \left(\frac{k^2+q^2+2kq\sinh qh \sinh kh - 2k^2\cosh qh \cosh kh}{k\sinh qh \cosh kh - q \sinh kh \cosh qh} \right),$$

$$G(\sigma,z',z) = \int_{\Gamma} \frac{ds}{2\pi i}\, e^{s\sigma} \tilde{G}(s,z',z),$$

$$\tilde{G}(s,z',z) = \frac{1}{\nu q}\left(\theta(z-z')\sinh q(z'-z) - \sinh qz' \cosh qz\right) + \frac{\sinh qz}{\nu q}\left(\frac{q\sinh kz' - \sinh qz'(k\cosh qh \cosh kh - q\sinh qh \sinh kh)}{k\sinh qh\cosh kh - q\sinh kh \cosh qh} \right),$$

where θ is the Heaviside function.

Appendix B. The function $\Delta(z)$

B.1. *Definition of the function*

The function $\Delta(z)$ is defined by equation (5.21) in an implicit form. Defining the function $\mathscr{H}(z,\Delta)$ as

$$\mathscr{H}(z,\Delta) = \frac{1}{\pi}\int_{\frac{1-z^2}{\Delta}}^{1} ds \left(\frac{\Delta s + z^2 - 1}{1-s^2} \right)^{1/2} - z, \tag{B 1}$$

the function $\Delta(z)$ is defined by $\mathscr{H}(z,\Delta)=0$ and is well defined only for $\Delta > |z^2-1|$ (this condition is related to $|E| < V_0$), but one always has a unique solution of (B 1) in that region. One can find directly that $\mathscr{H}(0,1)=0$ and then

$$\Delta(0) = 1. \tag{B 2}$$

For $z=1$ the equation

$$\mathscr{H}(1,\Delta) = \frac{1}{\pi}\int_0^1 ds \left(\frac{\Delta s}{1-s^2} \right)^{1/2} - 1 = 0, \tag{B 3}$$

gives

$$\Delta(1) = \frac{\pi^2}{\left(\int_0^1 ds \left(\frac{s}{1-s^2} \right)^{1/2} s \right)^2} \approx 6.88. \tag{B 4}$$

The derivate $\Delta(z)$ has the value

$$\Delta'(z) = -\frac{\partial_z \mathscr{H}}{\partial_\Delta \mathscr{H}} \tag{B 5}$$

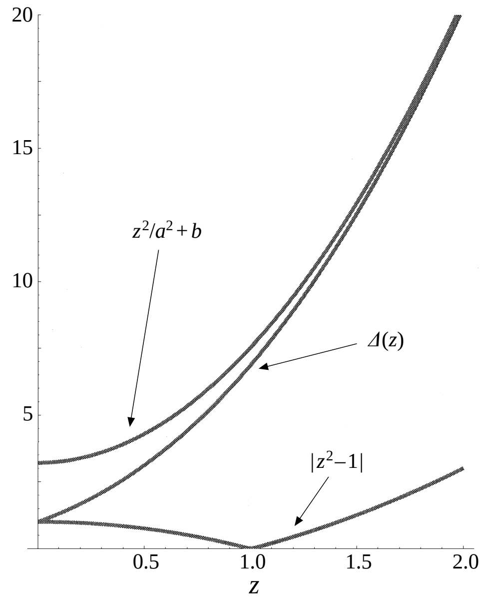

FIGURE 12. Drawing of the function $\Delta(z)$ and of the upper and lower bounds given in the text.

and from the definition of $\mathscr{H}$ one has

$$\partial_z \mathscr{H}(z,\Delta) = \frac{z}{\pi \Delta^{1/2}} \int_r^1 \mathrm{d}s \frac{1}{[(s-r)(1-s^2)]^{1/2}} - 1, \tag{B 6}$$

$$\partial_\Delta \mathscr{H}(z,\Delta) = \frac{1}{2\pi \Delta^{1/2}} \int_r^1 \mathrm{d}s \frac{s}{[(s-r)(1-s^2)]^{1/2}}, \tag{B 7}$$

where $r = (1-z^2)/\Delta$. For the point $z = 0$ where $\Delta = 1$ one can show from the previous expressions that

$$\Delta'(0) = 2\sqrt{2} \tag{B 8}$$

and also that $\Delta' > 0$, $\Delta'' > 0$.

B.2. *Asymptotic approximation*

Since $\Delta > |z^2 - 1|$ one can try to find a development of the form

$$\Delta(z) = z^2/a^2 + b(z), \tag{B 9}$$

where one expects that the first term is dominant for $z \gg 1$. This can be checked constructing a systematic expansion for $b(z)$. At lowest order we obtain from (B 1)

$$\mathscr{H}(z, z^2/a^2 + b(z))\big|_{z\gg 1} \approx \frac{z}{a\pi} \int_{-a^2}^1 \mathrm{d}s \left(\frac{s+a^2}{1-s^2}\right)^{1/2} - z \tag{B 10}$$

and equating this to zero is consistent only if one has a solution for the equation

$$a = \frac{1}{\pi} \int_{-a^2}^1 \mathrm{d}s \left(\frac{s+a^2}{1-s^2}\right)^{1/2}; \tag{B 11}$$

numerically we find the solution $a \approx 0.481$.

At the next order we have the first contribution to $b(z)$ and a Taylor development gives

$$\mathscr{H}(z, z^2/a^2 + b(z)) \approx \mathscr{H}(z, z^2/a^2) + \partial_\Delta \mathscr{H}(z, z^2/a^2) b(z) + O(b(z)^2). \tag{B 12}$$

Since we expect that $b(z)$ is small we truncate and obtain

$$b(z) \approx -\frac{\mathscr{H}(z, z^2/a^2)}{\partial_\Delta \mathscr{H}(z, z^2/a^2)}. \tag{B 13}$$

Notice that $\mathscr{H}(z, z^2/a^2)$ vanishes only if z is big enough and a perturbative calculation leads to

$$\mathscr{H}(z, z^2/a^2) \approx -\frac{a}{2\pi z} \int_{-a^2}^{1} \mathrm{d}s \frac{1}{[(s + a^2)(1 - s^2)]^{1/2}}, \tag{B 14}$$

which vanishes when $z \to \infty$. A similar calculation gives

$$\partial_\Delta \mathscr{H}(z, z^2/a^2) \approx \frac{a}{2\pi z} \int_{-a^2}^{1} \mathrm{d}s \frac{s}{[(s + a^2)(1 - s^2)]^{1/2}} \tag{B 15}$$

and finally we obtain, replacing (B 14) and (B 15) in (B 13), the value of $b(z)$

$$b(z) \approx b = \frac{\displaystyle\int_{-a^2}^{1} \mathrm{d}s \frac{1}{[(s + a^2)(1 - s^2)]^{1/2}}}{\displaystyle\int_{-a^2}^{1} \mathrm{d}s \frac{s}{[(s + a^2)(1 - s^2)]^{1/2}}}; \tag{B 16}$$

numerically we obtain $b \approx 3.21$. In figure 12 we compare the exact function $\Delta(z)$ calculated numerically with the asymptotic approximation $\Delta(z) \approx z^2/a^2 + b$ and with the lower bound $\Delta(z) \geqslant |z^2 - 1|$.

REFERENCES

Bechhoefer, J., Ego, V., Manneville, S. & Johnson, B. 1995 An experimental study of the onset of parametrically pumped surface waves in viscous fluids. *J. Fluid Mech.* **288**, 325–350.

Benjamin, T. B. & Ursell, F. 1954 The stability of the plane free surface of a liquid in vertical periodic motion. *Proc. R. Soc. Lond.* A **225**, 505–515.

Bergé, P., Pomeau, Y. & Vidal, C. 1984 Oscillateur libre-oscillateur amorti. In *L'ordre dans le Chaos*. Hermman.

Besson, T., Edwards, W. S. & Tuckerman, L. S. 1996 Two-frequency parametric excitation of surface waves. *Phys. Rev.* E **54**, 507–513.

Beyer, J. & Friedrich, R. 1995 Faraday instability: Linear analysis for viscous fluids. *Phys. Rev.* E **51**, 1162–1168.

Cerda, E. & Tirapegui, E. 1997*a* Faraday's instability for viscous fluids. *Phys. Rev. Lett.* **78**, 859–863.

Cerda, E. & Tirapegui, E. 1997*b* On the linear evolution of the surface of a viscous incompressible fluid. *Bull. l'Acad. R. Sci. de Belgique* (Classe des Sciences 7/12/1996).

Chandrasekar, H. 1981 *Hydrodynamic and Hydromagnetic Stability,* chap. 10. Dover.

Chen, P. & Viñals, J. 1997 Pattern selection in Faraday waves. *Phys. Rev. Lett.* **14**, 2670–2673.

Ciliberto, S. & Gollub, J. P. 1985 Phenomenological model of chaotic mode competition in surface waves. *Nuovo Cimento* **6D**, 309–316.

Douady, S. 1989 Instabilite parametrique d'ondes de surface. PhD thesis, Paris VI, Paris.

Douady, S. & Fauve, S. 1988 Pattern selection in Faraday instability. *Europhys. Lett.* **6**, 221–226.

Edwards, W. S. & Fauve, S. 1992 Parametric instability of a liquid-vapor interface close to the critical point. *Phys. Rev. Lett.* **68**, 3160–3164.

Edwards, W. S. & Fauve, S. 1993 Parametrically excited quasicrystalline surface waves. *Phys. Rev.* E **47**, 788–791.

Edwards, W. S. & Fauve, S. 1994 Patterns and quasi-patterns in the Faraday experiment. *J. Fluid Mech.* **278**, 123–148.

Eisenmenger, W. 1959 Dynamics properties of the surface tension of water and aqueous solutions of surface active agents with standing capillary waves in the frequency range 10 kc/s to 1.5 Mc/s. *Acustica* **9**, 328–340.

Elphick, C. & Meron, E. 1989 Localized structures in surface waves. *Phys. Rev.* A **40**, 3226–3229.

Gennes, P. G. de 1985 Wetting: statics and dynamics. *Rev. Mod. Phys.* **57**, 827–863.

Goldman, I. I. & Krivchenkov, V. D. 1961 *Problems in Quantum Mechanics*. Adison Wesley.

Kudrolli, A. & Gollub, J. P. 1996 Pattern and spatiotemporal chaos in parametrically forced surface waves: a systematic survey at large aspect ratio. *Physica* D **97**, 133–154.

Kumar, K. 1996 Linear theory of Faraday instability in viscous liquids. *Proc. R. Soc. Lond.* A **452**, 1113–1126.

Kumar, K. & Tuckerman, L. S. 1994 Parametric instability of the interface between two fluids. *J. Fluid Mech.* **279**, 49–68.

Landau, L. D. & Lifshitz, E. M. 1970 Parametric resonance. In *Mechanica*. Reverté.

Landau, L. D. & Lifshitz, E. M. 1987 *Fluid Mechanics*. Pergamon.

Lioubashevsky, O., Arbell, H. & Fineberg, J. 1996 Dissipative solitary states in driven surface waves. *Phys. Rev. Lett.* **76**, 3959–3963.

Lioubashevsky, O., Fineberg, J. & Tuckerman, L. S. 1997 Scaling of the transition to parametrically driven surface waves in highly dissipative systems. *Phys. Rev.* E **55**, 3832–3835.

Markushevich, A. I. 1970 Continuación Analítica en una Estrella. In *Teoría de las Funciones Analíticas*. Vol. 2. Mir.

Melo, F., Umbanhowar, P. & Swinney, H. L. 1994 Transition to parametric wave patterns in a vertically oscillated granular layer. *Phys. Rev. Lett.* **72**, 172–176.

Melo, F., Umbanhowar, P. & Swinney, H. L. 1995 Hexagons, kinks, and disorder in oscillated granular layers. *Phys. Rev. Lett.* **75**, 3838.

Messiah, A. 1961 *Quantum Mechanics*, Vol. 1. North Holland.

Morse, P. M. & Feshbach, H. 1953 *Methods of Theoretical Physics*, Vol. 1, Chap. 5. McGraw Hill.

Müller, H. W. 1993 Periodic triangular patterns in the Faraday experiment. *Phys. Rev. Lett.* **71**, 3287–3291.

Müller, H. W., Wittmer, H., Wagner, C., Albers, J. & Knorr, K. 1997 Analytic stability theory for Faraday waves and the observation of the harmonic surface response. *Phys. Rev. Lett.* **78**, 2357–2360.

Pushkarev, A. N. & Zakharov V. E. 1996 Turbulence of capillary waves. *Phys. Rev. Lett.* **76**, 3320–3324.

Schröder, E., Andersen, J. S., Levinsen, M. T., Alstrom, P. & Goldburg, W. I. 1996 Relative particle motion in capillary waves. *Phys. Rev. Lett.* **76**, 4717–4721.

Sorokin, V. I. 1957 The effects of fountain formation at the surface of a vertically oscillating liquid. *Sov. Phys. Acoust.* **3**, 281–295.

Umbanhowar, P., Melo, F. & Swinney, H. L. 1996 Localized excitations in a vertically vibrated granular layer. *Nature* **382**, 793.

Wu, J., Keolian, R. & Rudnick, I. 1984 Observation of a nonpropagating hydrodynamic soliton. *Phys. Rev. Lett.* **52**, 1421–1425.

Zhang, W. & Viñals, J. 1996 Square patterns and quasipatterns in weakly damped Faraday waves. *Phys. Rev.* E **53**, 4283–4286.

Zhang, W. & Viñals, J. 1997 Pattern formation in weakly damped parametric surface waves. *J. Fluid Mech.* **336**, 301–330.